Probing an untouchable environment as a resource for quantum computing

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The full control of many-body quantum systems is expected to be a key towards the future nano- and quantum technologies. Among others, the realisation of quantum information processing [1] has been studied intensively as a good test bed of quantum control as well as an ultimate engineering task that makes full use of quantum mechanical effects [2, 3]. Yet, manipulating quantum states is extremely hard, since information encoded in quantum states easily leaks out to the environment due to the inevitable interactions with it. In the theory of open quantum systems, an environment is usually treated as a large bath [4, 5], assuming that most of its dynamical details are averaged out. Thus we do not see it as a quantum object that we can control actively. In realistic situations, it is indeed near-impossible to precisely identify the quantum nature of an environment, let alone to control it at will. However, here we demonstrate how this formidable task can be achieved, provided the dimension of the environment can be regarded as finite within the timescale we manipulate the system. The information thereby obtained will be useful not only for deeper understanding of the system dynamics under decoherence but also for exploiting the environment as a *resource* for quantum engineering, such as quantum computation.

Specifically, our primary goal here is to identify the Hamiltonian H_{SE}, which describes the effective interaction between the principal system S and the surrounding system E and (a part of) its internal dynamics. The dimension d_E of E is also unknown a priori and is a subject of our identification. We shall call E the *environment* symbolically throughout the paper. The knowledge of H_{SE} is vital for controlling E as a useful system, not to mention for checking its controllability. Readers may be reminded of the methods of quantum process tomography (QPT)[1, 6–11] as means to determine all the parameters that characterise a general quantum evolution, namely a completely positive (CP) map. Nevertheless, QPT is a scheme to estimate the CP map for a quantum system for which we can prepare a specific state and perform measurements. Thus, the conventional QPT methods do not reveal the nature of an environment, which is beyond the reach of our measurement.

Incidentally, there have been a series of studies on Hamiltonian identification of a many-body system under the condition of limited access [12–15]. However, all of them assume that a priori knowledge was available about the system configuration and the initialisability of the system state that depends on the type of interaction. In the present analysis, no particular assumptions are made about system structures or the type of interaction, except for the finiteness of d_E. Therefore, the task is even more nontrivial than existing tomographic schemes.

The basic idea is as follows. We first steer the joint system SE and an ancillary system A into a maximally entangled state with respect to the partition between SE and A, while S and E keep interacting through H_{SE}. Then, the entire state on S, E, and A will be a product of two maximally entangled pairs of subsystems, which are S and $A_1 \subset A$, and E and $A_2 (= A \backslash A_1)$. The time evolution of SE, expressed by an operator $U_{SE}(t) = \exp(-iH_{SE}t)$, can be observed as a *mirror image* on the side of A_1 and A_2, due to a property of maximally entangled pairs. The desired information on the Hamiltonian H_{SE} can be extracted by performing QPT on S and A.

We shall primarily focus on the ideal case where the effect of errors is negligible, in order to elucidate the essence of the idea. Particularly, the consideration of perfect entanglement between SE and A allows us to rigorously prove the propositions necessary for the main results.

Let $\mathcal{H}_S$ and $\mathcal{H}_E$ be the Hilbert spaces of the principal system (S) and its environment (E), whose dimensions are d_S and d_E, respectively. A key assumption we make here is that d_E is finite, although it can be unknown. Naturally, d_E may be infinitely large in general, but we consider a situation where the system S effectively interacts with only a finite dimensional subspace E of the *universe* E'. That is, the interaction between S and E is so dominant within the relevant timescale for describing the dynamics of the system that we can justify this assumption. In other words, the combined system in $\mathcal{H}_S \otimes \mathcal{H}_E$ undergoes a unitary evolution.

For longer timescales, the combined system SE cannot be immune to the effect of interactions with its surrounding environment E'. A state ρ_{SE} on $\mathcal{H}_S \otimes \mathcal{H}_E$ is now subject to equilibration and will tend to some fixed state $\rho_{SE}^{(0)}$. This fact can be used to reset the state ρ_{SE} before iterating the protocol. As will be clear later, the initial state $\rho_{SE}^{(0)}$ can be arbitrary, as long as it is a fixed state, albeit unknown.

Another important assumption is that we are allowed to provide as many ancillary states as necessary, each of which is a maximally entangled pair between a_1 and a_2:

$$|\Upsilon_{a_1 a_2}\rangle = \frac{1}{\sqrt{d_S}} \sum_{i=1}^{d_S} |i_{a_1} i_{a_2}\rangle. \tag{1}$$

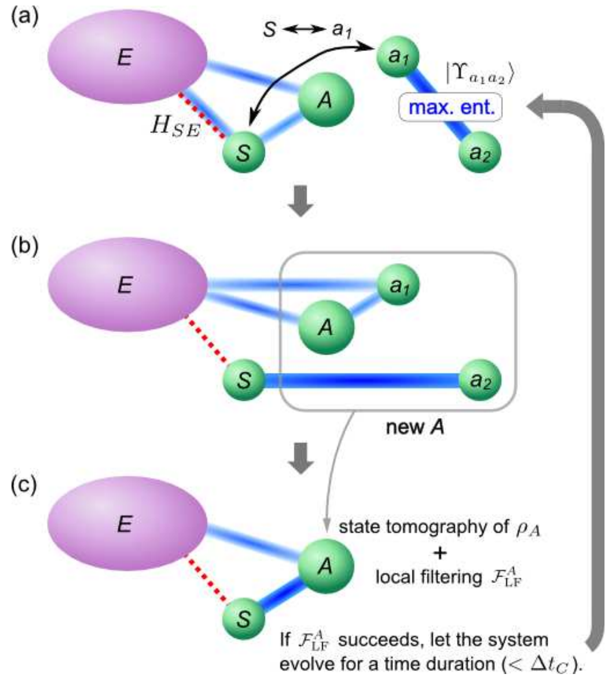

FIG. 1: The protocol for steering the state on SEA towards the maximally entangled one between SE and A. The thick blue lines and the red dotted line represent entanglement and the interaction, respectively. The three systems, S, E, and A, are in some entangled state after foregoing rounds of the protocol, while A is a null space for the first round. In (a), a maximally entangled pair $|\Upsilon_{a_1a_2}\rangle$ is provided and the states of the subsystem a_1 and S are swapped to make the entanglement network look like (b). Relabeling A, a_1, and a_2 as a new A as in (c), we perform state tomography of A and subsequently a local filtering operation $\mathcal{F}^A_{\rm LF}$ on A. If $\mathcal{F}^A_{\rm LF}$ succeeds, iterate the procedure, going back to (a).

They will form the ancillary system A as the protocol proceeds. Also, the interaction between A and E is assumed to be negligible. Further, we shall take it for granted that ρ_{SE} is initially pure, because, otherwise, we can always purify it by appending an additional Hilbert space.

In order to make use of the *'mirroring effect'* of entanglement to identify H_{SE}, we first need to steer the state on SEA to establish maximal entanglement between SE and A. Let us describe how the state-steering protocol goes, and delineate why it works out for our purpose. Figure 1 depicts the state-steering protocol. We start with an initial (fixed, but unknown) state $\rho^{(0)}_{SE}$ and abundant copies of $|\Upsilon_{a_1a_2}\rangle$ in Eq. (1). At $t = 0$, $\mathcal{H}_A$ is a null space, supporting no states. The SWAP operation between S and a_1, which must be fast enough compared with the system dynamics, will be denoted as SWAP_{Sa_1}. The C-th round of the protocol proceeds as follows (C starts from zero):

Step 1: Apply SWAP_{Sa_1}, where a_1 is a subsystem of the newly provided $|\Upsilon_{a_1a_2}\rangle$, and then let A incorporate a_1 (the former S) and a_2.

Step 2: Perform state tomography of ρ_A, and apply a local filtering operation $\mathcal{F}^A_{\rm LF}$ on ρ_A. If it fails, carry out the whole protocol from the beginning.

Step 3: Let the SE system evolve for a time duration ($< \Delta t_C$) so that the functional of ρ_{SA}, ΔE_{SA}, which is defined below by Eq. (2), increases by $\epsilon_C > 0$. See Sec. IV of the supplementary information (SI) as to how we should determine Δt_C and ϵ_C.

Step 4: Terminate the protocol if ΔE_{SA} is found to be non-increasing; otherwise, let the SE system evolve so that $\Delta E_{SA} \geq \epsilon_C$, and go back to Step 1.

Intuitively, Δt_C and ϵ_C are set so that we can complete the steering protocol within the desired time period, which can be made as short as possible at the expense of success probability.

The local filtering operation on ρ_A is written as $\mathcal{F}^A_{\rm LF}\rho_A = F_{\rm LF}\rho_A F^\dagger_{\rm LF}$, where $F_{\rm LF} = \sqrt{\lambda_{\min}\cdot\rho_A^{-1}}$ with ρ_A^{-1} the inverse of ρ_A on its support and $\lambda_{\min}$ the smallest eigenvalue of ρ_A. The probability of success of $\mathcal{F}^A_{\rm LF}$ is $\lambda_{\min}\cdot \mathrm{rank}\rho_A$.

The quantity ΔE_{SA} we measure in Step 3 is defined as

$$\Delta E_{SA} := S(\rho_{SA}) - S(\rho_A) + \ln d_S, \qquad (2)$$

where $S(\rho) = -\mathrm{Tr}(\rho\ln\rho)$ is the von Neumann entropy and $\rho_A = \mathrm{Tr}_S\rho_{SA}$. While what it represents may not be obvious at first sight, ΔE_{SA} is the change in entanglement between SA and E due to SWAP_{Sa_1} in Step 1 of the following round. To see this, letting $E(|\psi\rangle)$ be the amount of entanglement in state $|\psi\rangle$ with the partition $SE|A$, we have

$$\begin{aligned}\Delta E_{SA} &= E(|\Psi'_{SEA'}\rangle) - E(|\Psi_{SEA}\rangle)\\ &= E(|\Psi_{EAa_1}\rangle\otimes|\Upsilon_{Sa_2}\rangle) - E(|\Psi_{SEA}\rangle\otimes|\Upsilon_{a_1a_2}\rangle)\\ &= S(\rho_{SA}) - S(\rho_A) + \ln d_S, \end{aligned} \qquad (3)$$

where A' is the relabeled A, i.e., Aa_1a_2.

We can rewrite ΔE_{SA} as

$$\Delta E_{SA} = D(\rho_{SE}||\rho_S\otimes\rho_E) + D(\rho_S||I^S/d_S), \qquad (4)$$

where $D(\rho||\sigma) = \mathrm{Tr}(\rho\ln\rho - \rho\ln\sigma)$ is the relative entropy between ρ and σ, and I^S is a $d_S\times d_S$ identity matrix. Equation (4) shows $\Delta E \geq 0$ due to the nonnegativity of the relative entropy. Thus, we can see that, when ΔE is small, ρ_{SE} and ρ_S are very close to the product state $\rho_S\otimes\rho_E$ and the completely mixed state I^S/d_S, respectively.

The above observation implies that when $\Delta E_{SA} = 0$ there is a subsystem A_1 of A that is maximally entangled with S. Further, we expect that the remaining part A_2 of A is maximally entangled with E as a result of $\mathcal{F}^A_{\rm LF}$. This naive guess is proven in detail in the SI.

Note, however, that the argument in the SI involves some mathematical subtleties concerning multiple possibilities of the set $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ that leads to identical observable dynamics on S and A. We call this set of

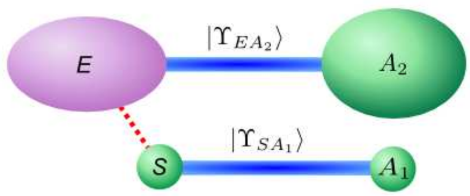

FIG. 2: The desired state of S, E, and A. A subsystem A_1 of A and its complement A_2 are maximally entangled with S and E, respectively. Further executions of the entangling protocol do not change the entanglement structure: They only enlarge the size of A_2 without increasing the entanglement with E.

three ingredients a *triple*. More precise propositions we show in the SI are as follows:

- There can be equivalent classes of triples, so that all triples within a single equivalent class give rise to the identical observable dynamics on S and A, regardless of any active controls on them. For our purpose of controlling E, it suffices to identify one in the class for the observed time evolution on SA.

- The resulting state $|\Psi_{SEA}\rangle$ of the state-steering protocol satisfies a condition, which is expressed as

$$\mathrm{Tr}_E(|\Psi_{SEA}\rangle\langle\Psi_{SEA}|) = |\Upsilon_{SA_1}\rangle\langle\Upsilon_{SA_1}| \otimes \rho_{A_2}, \quad (5)$$

  where A_1 and A_2 are non-overlapping subspaces of A such that $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, $|\Upsilon_{SA_1}\rangle$ is a maximally entangled state, and ρ_{A_2} is a projector onto $\mathcal{H}_{A_2}$. We refer to Eq. (5) as the maximal entanglement (*ME*) condition (Sec. IV of SI).

- If the state $|\Psi_{SEA}\rangle$ fulfills the ME condition, then there is a maximal entanglement between systems SE and A. That is, there exists an equivalent triple, in which A_2 is fully entangled with the entire E. Thus

$$|\Psi_{SEA}\rangle = |\Upsilon_{SA_1}\rangle \otimes |\Upsilon_{EA_2}\rangle, \quad (6)$$

  as depicted in Fig. 2 (see Sec. II of SI).

Figure 3 illustrates an intuitive description of the equivalence, which is mentioned in the first statement, between dynamics observed on a subsystem of a larger system. After defining the equivalence classes and the ME condition (in the second statement), we show that, once the ME condition is fulfilled, observing the natural time evolution of SA, i.e., without active controls on them, is sufficient to specify the class (Theorem 1 in SI). The third statement is to ensure that even if the support of ρ_{SE} in $\mathcal{H}_E$ may move around, it is still possible to find a Hamiltonian H_{SE} with which ρ_{SE} can be seen as a stationary state on SEA (Theorem 2 in SI). Then, this justifies the use of Eq. (7) below, or the mirroring effect of maximally entangled states, for our Hamiltonian tomography.

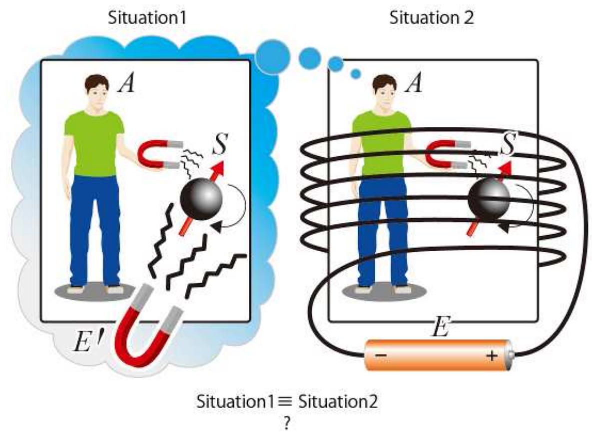

FIG. 3: An illustration of the physical situations that lead to identical observable dynamics. In this figure, the experimenter measures the dynamics a small magnet (spin) shows by varying local control parameters. There could be multiple possibilities of external elements, i.e., environment, which would give rise to the same dynamics of the magnet, no matter what he controls. For example, he cannot distinguish two situations; the whole laboratory may be in the magnetic field generated by wire that goes around the lab, or the external field of the same strength may be due to a strong permanent magnet. Similarly, in the case written in the main text, there could be many triples $\{(d_E, |\Psi_{SEA}\rangle, H_{SE})\}$ that lead to indistinguishable dynamics on SA.

Now that we have two pairs of maximally entangled states, we move on to the Hamiltonian identification stage. One of the well-known properties of maximally entangled states is that an application of unitary operation on two subsystems at one side of the pairs is equivalent to that of its transpose on the other side. For $|\Psi_{SEA}\rangle$ in Eq. (6), we can write

$$U_{SE}|\Psi_{SEA}\rangle = V_A|\Psi_{SEA}\rangle, \quad (7)$$

where $V_A = U_{SE}^T$ acting on $\mathcal{H}_A$. Therefore, the unitary evolution we observe on the ancillary system $A = A_1A_2$ should contain information about the Hamiltonian $H_{SE} = i/t \ln U_{SE}(t)$. Naturally, however, simply looking at the state of A does not reveal any information on U_{SE}, but it turns out that identifying $\rho_{SA}(t)$ by quantum state tomography suffices for our purpose. The origin of time t may be reset to the instance where the state-steering protocol is completed.

Let us describe a specific method of estimating the Hamiltonian H_{SE} after preparing the state $|\Psi_{SEA}\rangle$ in Eq. (6). As all we can probe is systems A and S, we can only observe the time evolution of the reduced density operator $\rho_{SA}(t) = \mathrm{tr}_E|\Psi_{SEA}\rangle\langle\Psi_{SEA}|$. Our task is to find a Hamiltonian H_{SE} that generates the time evolution of $\rho_{SA}(t)$, such that

$$i\frac{\partial}{\partial t}\rho_{SA} = [H_{SE}^T \otimes I_S, \rho_{SA}], \quad (8)$$

where H_{SE}^T acts on $\mathcal{H}_A$, despite its notation. The state ρ_{SA} as a function of time can be obtained by iterating quantum state tomography at different times and is, thus a sort of inverse problem of the usual calculation of Eq. (8). Note that in order to correctly specify H_{SE}^T on the side of A as an interaction between two subsystems, the basis we employ needs to be written as a product of A_1 and A_2, which are entangled with S and E, respectively, at the instance when the state $|\Psi_{SEA}\rangle$ in Eq. (6) was realised.

Since the system A is driven by H_{SE}^T, which is finite-dimensional, $\rho_{SA}(t)$ can be Fourier transformed such that each component ρ_α evolves with a frequency $\theta_\alpha\,(\alpha = 0, 1, ..., L)$, where we set $\theta_0 = 0$ and L can be as large as $d_S d_E(d_S d_E - 1)/2$. Namely,

$$\rho_{SA}(t) = \rho_0 + \sum_{\alpha=1}^{L} \left(\rho_\alpha e^{i\theta_\alpha t} + \rho_\alpha^\dagger e^{-i\theta_\alpha t}\right), \tag{9}$$

where all θ_α and ρ_α can be obtained from the data of $\rho_{SA}(t)$. Substituting Eq. (9) into Eq. (8), we have

$$[H_{SE}^T, \rho_0] = 0 \tag{10}$$
$$\left[H_{SE}^T, \rho_\alpha\right] = \theta_\alpha \rho_\alpha \tag{11}$$
$$\left[H_{SE}^T, \rho_\alpha^\dagger\right] = -\theta_\alpha \rho_\alpha^\dagger. \tag{12}$$

Now we choose a set of orthogonal bases of hermitian operators on $\mathcal{H}_A \otimes \mathcal{H}_S$ as $\{h_j\}_{j=1}^{d_S^2 d_E}$ so that any hermitian operators on the same Hilbert space can be written as a linear combination of h_j. The relevant hermitian operators in our context are ρ_0, $\rho_\alpha + \rho_\alpha^\dagger$, $i(\rho_\alpha - \rho_\alpha^\dagger)$ and $H_{SE}^T \otimes I_S$. If we write the last one as

$$H_{SE} \otimes I_S = \sum_j \kappa_j h_j, \tag{13}$$

Hamiltonian tomography can be seen as a task of identifying coefficients κ_j. Expressing the above hermitian operators in terms of h_j and using Eqs. (10)-(12), we arrive at a system of linear equations with respect to κ_j. The solution of the resulting set of equations would be non-unique. However, once we obtain a possible set of $\{\kappa_j\}$, it suffices to describe all the dynamics we observe through the accessible systems S and A, if any. Hence, the task of Hamiltonian tomography is completed. For the full description, refer to Sec. III of SI.

The Hamiltonian H_{SE} thereby estimated contains all the necessary information to characterise the observable dynamics, albeit unmodulable per se. What we can control actively is the system S. Thus, the dynamics of the entire system SE is governed by the Hamiltonian

$$H(t) = H_{SE} + \sum_n f_n(t) H_S^{(n)}, \tag{14}$$

where $H_S^{(n)}$ are independent Hamiltonians that act on S and can be modulated by $f_n(t)$. As we have already identified H_{SE}, there is sufficient information to judge the controllability of the system SE under this Hamiltonian (14). A theorem from the quantum control theory states that the set of realisable unitary operations on the system is generated by dynamical Lie algebra [16–18]. Dynamical Lie algebra can be computed by taking all possible (repeated) commutators of operators in Eq. (14), i.e., iH_{SE} and $\{iH_S^{(n)}\}$, and their real linear combinations.

Therefore, our knowledge of H_{SE} allows the controllable system to encompass not only the principal system S but also (a part of) the environment E. That is, we are now able to exploit the dynamics inside E for useful quantum operations, such as quantum computing, by controlling a small system S only. This is the same situation as in refs. [19–23], where only a small subsystem is accessed to control a large system.

We have shown the possibility of probing a large surrounding quantum system (environment) through a small principal system despite the lack of direct accessibility, provided the environment is effectively finite-dimensional. By probing, we mean fully identifying the Hamiltonian and utilising it as a useful resource for quantum control, e.g., quantum computation. This goes strongly against our intuitive supposition that environment is usually a villain with respect to the protection of quantum coherence and that the Hamiltonian H_{SE} is not estimable. Although our method might not be realisable in the lab straightaway, it opens up a path to the novel exploitation of the environment for quantum engineering.

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Supplementary information: Probing an untouchable environment as a resource for quantum computing

I. PROBLEM SETTINGS AND EQUIVALENCE OF DYNAMICAL BEHAVIOURS

A. Problem settings

We consider a joint system consisting of three parts: the principal system S, its surrounding system (environment) E, and an ancillary system A, whose Hilbert spaces are denoted as $\mathcal{H}_S$, $\mathcal{H}_E$, and $\mathcal{H}_A$, respectively. The entire system SEA on $\mathcal{H}_S \otimes \mathcal{H}_E \otimes \mathcal{H}_A$ is a non-dissipative closed system. The principal system S interacts with its environment E via Hamiltonian H_{SE}, while E does not interact directly with the ancillary system A. Thus, the Hamiltonian of the joint system can be written as $H_{SE} \otimes I_A$. We are not allowed to access the environmental system E directly, which means that no part of E can be a subject of direct control or measurement. Meanwhile, we are able to perform any quantum operations and measurements on the joint system SA instantaneously.

A key assumption we make is that the environmental system E is finite-dimensional, i.e., $d_E := \dim E < +\infty$. Those systems under our control, A and S are also finite-dimensional, and naturally $d_A := \dim \mathcal{H}_A$ and $d_S := \dim \mathcal{H}_S$ are always known. We do not assume any prior knowledge of d_E, the interaction Hamiltonian H_{SE}, and the state on $\mathcal{H}_{SEA} := \mathcal{H}_S \otimes \mathcal{H}_E \otimes \mathcal{H}_A$. For most of the discussion in this supplementary information (SI), d_A refers to the dimension of A after the state-steering protocol (see the main text and Sec IV in SI) has been completed, unless stated otherwise.

Under these settings, our goal is to obtain as much information as possible about E and the interaction between S and E, namely, $\mathcal{H}_E$, H_{SE}, and the state on $\mathcal{H}_{SEA}$. A central tool for the information acquisition is quantum state tomography [1] of the joint system SA to determine ρ_{SA} as a function of time. Hence, we assume that it is possible to prepare an identical, but not necessarily known, initial state $|\Psi_{SEA}\rangle$ on the whole system $\mathcal{H}_{SEA}$ as many times as necessary. Pragmatically, such a state initialisation can be achieved by waiting for the equilibration of the state, which is caused by the interaction with a larger environmental system that surrounds E [2, 3]. The timescale for such an equilibration is much longer than the one within which the entire system of SEA can be considered closed. The initial time t_0 is defined as the time when the state initialisation is completed.

Since we have prior information about d_A and d_S, our system dynamics can be characterised by d_E, a state on $\mathcal{H}_{SEA}$ at the time t_0, and the Hamiltonian H_{SE}. Note that at t_0, we can always assume the initial state on $\mathcal{H}_{SEA}$ is pure. This is because when the state on $\mathcal{H}_{SEA}$ is mixed, we can append an extra Hilbert space $\mathcal{H}_{E'}$ to the system so that the state on $\mathcal{H}_{SEA} \otimes \mathcal{H}_{E'}$ is pure [3, 4]. Then, we simply redefine $\mathcal{H}_E \otimes \mathcal{H}_{E'}$ as $\mathcal{H}_E$, and $H_{SE} \otimes I_{E'}$ as a new Hamiltonian H_{SE} to restart the whole discussion. Therefore, we need a set of three elements $(d_E, |\Psi_{SEA}\rangle, H_{SE})$, which we shall call a *triple*, to characterise the behaviour of our system under the effect of environment between the times t_0 and t_∞ ($t_0 < t_\infty$). Although we choose $t_\infty = +\infty$, which is theoretically natural, since we practically perform an experiment within a finite time length, the consideration of a finite t_∞ is also useful as we will see later.

B. The equivalence due to indistinguishable dynamics

As mentioned above, our primary goal is to identify the triple $(d_E, |\Psi_{SEA}\rangle, H_{SE})$. Yet, it appears to be a formidable, if not impossible, task to completely specify the triple when our access is limited to only S and A. What if there are more than one possible triple? In the context of system identification, the problem of indistinguishable system models for a given experimental data set has been studied in the classical setting for a long time [5] and also recently discussed in quantum scenario, in which the entire system is known to be controllable [6]. Similarly to these cases, for our task of identifying the system E, it turns out that even if there were multiple possibilities of triples that give rise to the same dynamical behaviour on SA, the differences between them would not lead to distinct outcomes of quantum control [7] of E through S. In other words, if there were two indistinguishable environmental systems characterised by $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ and $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ in the time period $[t_0, t_\infty)$, the results of any quantum computation that utilises E as a resource would be independent of whether the *true* environment was either of them. Therefore, for the acquisition of information on the environmental system $\mathcal{H}_E$ towards the exploitation of E as a (partial) resource for quantum computing, it suffices to determine the equivalence class on the set of all triples through all possible sequences of quantum operations on $\mathcal{H}_S \otimes \mathcal{H}_A$.

Before giving a rigorous definition of the equivalence class of triples, let us first specify all operations we can apply on the system. First, we do not consider operations that are applied continuously in time. Thus, what we consider to be applicable is a sequence of instantaneous quantum operations [3, 4, 8, 9], $\{\Gamma_i\}_{i=1}^n$ at time t_i, where $n < +\infty$ and $t_i < t_j$ for all $i < j$. Second, a quantum operation Γ_i can be non-deterministic, i.e., trace non-increasing, because we can always post-

select the measurement results. Third, we are allowed to append and remove finite-dimensional ancillary systems, which means that Γ_i is a quantum operation on $\mathfrak{B}\left(\mathcal{H}_{A_{i-1}} \otimes \mathcal{H}_S\right)$ to $\mathfrak{B}\left(\mathcal{H}_{A_i} \otimes \mathcal{H}_S\right)$, where $\mathfrak{B}\left(\mathcal{H}\right)$ is a linear space of all (bounded) linear operators on $\mathcal{H}$, $\mathcal{H}_{A_{i-1}} \neq \mathcal{H}_{A_i}$ in general, $\dim \mathcal{H}_{A_i} < +\infty$, and $\mathcal{H}_{A_0} := \mathcal{H}_A$.

Hence, our definition of the equivalence class is as follows:

Definition 1. ***(The equivalence between triples)*** *A triple $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ is said to be equivalent to another triple, $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$, in $[t_0, t_\infty)$, if they satisfy*

$$\begin{aligned} &\mathrm{Tr}_E \left(\prod_{i=1}^{n} (\Gamma_i \otimes \mathcal{I}_E) \circ \left(\mathcal{I}_A \otimes \mathcal{U}_{SE}^{(i)}\right) P(|\Psi_{SEA}\rangle) \right) \\ =&\mathrm{Tr}_E \left(\prod_{i=1}^{n} (\Gamma_i \otimes \mathcal{I}_E) \circ \left(\mathcal{I}_A \otimes \tilde{\mathcal{U}}_{SE}^{(i)}\right) P(|\tilde{\Psi}_{SEA}\rangle) \right) \end{aligned} \tag{S1}$$

for all $n \in \mathbb{N}$ and all sequences of completely positive trace non-increasing maps $\{\Gamma_i\}_{i=1}^n$ [3, 4, 8, 9]. Each Γ_i is a map from $\mathfrak{B}\left(\mathcal{H}_{A_{i-1}} \otimes \mathcal{H}_S\right)$ to $\mathfrak{B}\left(\mathcal{H}_{A_i} \otimes \mathcal{H}_S\right)$ and performed at time t_i ($i \in \{1, 2, ..., n\}$). We will denote the equivalence between triples as $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$.

In Eq. (S1), $\mathcal{I}_A$ and $\mathcal{I}_E$ are identity (super)operators on $\mathcal{H}_A$ and $\mathcal{H}_E$, respectively. Also, $P(|\Psi\rangle)$ stands for $P(|\Psi\rangle) := |\Psi\rangle\langle\Psi|$. $\mathcal{U}_{SE}^{(i)}$ is given as

$$\mathcal{U}_{SE}^{(i)}(\rho) := \exp\left(-iH_{SE}(t_i - t_{i-1})\right) \rho \exp\left(iH_{SE}(t_i - t_{i-1})\right),$$

and $\tilde{\mathcal{U}}_{SE}^{(i)}(\rho)$ is defined similarly with $\tilde{H}_{SE}$ instead of H_{SE}. We can easily see that the relation "$\equiv$" is reflective, symmetric, and transitive. Thus, it is an equivalence relation in the mathematical sense [10], and a set of all the triples can be decomposed into equivalence classes accordingly.

Our definition of equivalence here differs from the one in [6] in that ours includes the possibility of appending an arbitrarily large ancillary system. Also, in Def. 1 above, the controllability of the system of interest is not assumed. Such a consideration is important especially when we can utilise the joint system SE as a part of a larger quantum network, rather than an isolated quantum computer.

Let us slightly simplify the definition of the above equivalence relation for the following discussion. Here, we define a Hilbert space A' that includes all A_i as its subspace.

Lemma 1. *A triple $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ is equivalent to anther one $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ in $[t_0, t_\infty)$, if and only if they satisfy*

$$\begin{aligned} &\mathrm{Tr}_E P\left(\left(\prod_{i=1}^{n} N_{SA'}^{(i)} \cdot U_{SE}^{(i)} \right) |\Psi_{SEA}\rangle \right) \\ =&\mathrm{Tr}_E P\left(\left(\prod_{i=1}^{n} N_{SA'}^{(i)} \cdot \tilde{U}_{SE}^{(i)} \right) |\tilde{\Psi}_{SEA}\rangle \right) \end{aligned} \tag{S2}$$

for all $n \in \mathbb{N}$, all sequences of real numbers $\{t_i\}_{i=1}^n$ with $t_i < t_j$ for $i < j$, all finite dimensional Hilbert spaces $\mathcal{H}_{A'}$ which includes $\mathcal{H}_A$ as a subspace ($\mathcal{H}_A \subset \mathcal{H}_{A'}$), and all sequences of operators $\{N_{SA'}^{(i)}\}_{i=1}^n$ on $\mathcal{H}_{A'} \otimes \mathcal{H}_S$.

In Eq. (S2), $U_{SE}^{(i)} := \exp\left[-iH_{SE}(t_i - t_{i-1})\right]$, and I_E and $I_{A'}$ are omitted. We shall not write identity operators explicitly throughout the paper when there is no risk of confusion.

(Proof)
In order to prove the "*only if*" part, since $\Gamma_i(\rho) := N_{SA'}^{(i)} \rho N_{SA'}^{(i)\dagger}$ is a trace non-increasing CP map, we simply need to define $\mathcal{H}_{A'}$ as $\mathcal{H}_{A'} := \mathcal{H}_{A_i}$. For the "*if*" part, we define $\mathcal{H}_{A'} := \bigoplus_{i=0}^{n} \mathcal{H}_{A_i}$. Then, the linearity of the partial trace Tr_E guarantees the statement of the lemma. ■

II. THE MAXIMAL ENTANGLEMENT CONDITION AND THE EQUIVALENCE BETWEEN TRIPLES

We now introduce a condition that is of crucial importance for our analysis. We shall refer to it as the maximal entanglement (ME) condition. It will be shown that when our system satisfies this condition, the natural time evolution of the system SA completely determines the equivalence class (the subsection II A). By natural time evolution, we mean the evolution of the system without active operations on it, namely, the evolution that is driven only by the system Hamiltonian, which is H_{SE} in our case. Then, we prove that when the system satisfies the ME condition, there exists an equivalent triple, in which E which is maximally entangled [3, 11, 12] with (a subset of) A (the subsection II B). We note that although our main focus is on a finite-dimensional $\mathcal{H}_E$ in this paper, some of the theorems and lemmas in this section are valid even for infinite-dimensional $\mathcal{H}_E$, as long as d_A and d_S are finite.

A. The maximal entanglement condition

The condition for the equivalence of triples still appears quite complicated even in the form of Lemma 1. In this subsection, we prove that the condition for the equivalence reduces to merely the indistinguishability of the natural time evolution of the system SA, when the condition defined in the following is satisfied:

Definition 2. ***(The maximal entanglement condition)*** *The maximal entanglement (ME) condition is said to be satisfied by a triple $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ in $[t_0, t_\infty)$, if for all $t \in [t_0, t_\infty)$, there exist Hilbert spaces $\mathcal{H}_{A_1}(t)$ and $\mathcal{H}_{A_2}(t)$ such that $\mathcal{H}_A = \mathcal{H}_{A_1}(t) \otimes \mathcal{H}_{A_2}(t)$, $\dim \mathcal{H}_S = \dim \mathcal{H}_{A_1}(t)$, and*

$$\mathrm{Tr}_E P\left(|\Psi_{SEA}(t)\rangle\right) = P\left(|\Upsilon_{SA_1}(t)\rangle\right) \otimes \rho_{A_2}(t), \tag{S3}$$

where $|\Upsilon_{SA_1}(t)\rangle$ is a maximally entangled state on $\mathcal{H}_S \otimes \mathcal{H}_{A_1}(t)$ and a state $\rho_{A_2}(t)$ is a projector onto $\mathcal{H}_{A_2}(t)$ (up to a proportionality constant).

Throughout this paper, $|\Upsilon_{XY}\rangle$ denotes a state that is maximally entangled *fully* on the space $\mathcal{H}_X \otimes \mathcal{H}_Y$ specified by the subscripts. That is, $|\Upsilon_{XY}\rangle = d_X^{-1/2} \sum_{i=1}^{d_X} |i_X\rangle|i_Y\rangle$, where $\{|i_X\rangle\}$ and $\{|i_Y\rangle\}$ are orthonormal bases of $\mathcal{H}_X$ and $\mathcal{H}_Y$, and $d_X = \dim\mathcal{H}_X (= d_Y)$.

In the above definition, $|\Psi_{SEA}(t)\rangle$ is the entire state at time t, i.e.,

$$|\Psi_{SEA}(t)\rangle := \exp(-iH_{SE}(t-t_0))|\Psi_{SEA}\rangle. \tag{S4}$$

Here, we note that the Hilbert spaces $\mathcal{H}_{A_1}(t)$ and $\mathcal{H}_{A_2}(t)$ may vary inside $\mathcal{H}_A$ as the time evolution of SE (due to H_{SE}) would be reflected in A_1 and A_2 through entanglement.

Equation (S3) implies the existence of a pure state $|\Phi_{EA_2}(t)\rangle$ on $\mathcal{H}_E \otimes \mathcal{H}_{A_2}(t)$ such that

$$|\Psi_{SEA}(t)\rangle = |\Upsilon_{SA_1}(t)\rangle \otimes |\Phi_{EA_2}(t)\rangle, \tag{S5}$$

where $|\Phi_{EA_2}(t)\rangle$ is a maximally entangled state (MES) on a subspace of $\mathcal{H}_E \otimes \mathcal{H}_{A_2}(t)$. Yet the rank of the reduced density matrix ρ_E may be smaller than the dimension of the system E: $\mathrm{rank}\rho_E < d_E$. It turns out, however, it is possible to choose a triple $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ (equivalent to $(d_E, |\Psi_{SEA}\rangle, H_{SE})$) so that $|\tilde{\Phi}_{EA_2}(t)\rangle$ can be expressed as $|\Upsilon_{EA_2}\rangle = \tilde{d}_E^{-1/2} \sum_{i=1}^{d_E} |i_{A_2}\rangle|i_E\rangle$, i.e., a state that is not only maximally entangled but also satisfies $\mathrm{rank}\rho_E = d_E$ (see Theorem 2 and Corollary 3). When the ME condition is found to be satisfied, we can take it for granted that the whole E is maximally entangled with $A_2 \subset A$, despite the inaccessibility of E.

An observation is that the fulfillment of the ME condition can be tested through tomography of the state on SA only. This fact leads to a lemma:

Lemma 2. *Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$, and $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ in $[t_0, t_\infty)$. Then, $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ also satisfies the ME condition in $[t_0, t_\infty)$.*

(Proof) Due to the definition of the equivalence, if $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ in $[t_0, t_\infty)$, we have, for all $t \in [t_0, t_\infty)$,

$$\mathrm{Tr}_E P\left(|\Psi_{SEA}(t)\rangle\right) = \mathrm{Tr}_E P\left(|\tilde{\Psi}_{SEA}(t)\rangle\right), \tag{S6}$$

where $|\Psi_{SEA}(t)\rangle$ is given in Eq. (S4) and $|\tilde{\Psi}_{SEA}(t)\rangle$ is defined similarly. Then, since the fulfillment of the ME condition only depends on the reduced density operator on $\mathcal{H}_S \otimes \mathcal{H}_A$ during $[t_0, t_\infty)$, Eq. (S6) guarantees the statement of the lemma. ■

Lemma 2 implies that the ME condition can also be considered as a property of equivalence class of triples.

Let us now present a theorem, which claims that if two triples give rise to an identical (natural) time evolution on SA and if one of them satisfies the ME condition, then those two triples are equivalent.

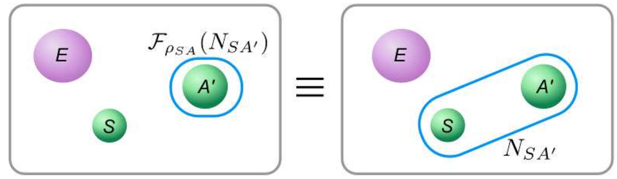

FIG. 4: Schematic illustration of the relation in Eq. (S9). System A' includes A_1 and A_2 as its subsystems. No matter what operation $N_{SA'}$ is performed on SA', the same effect can be realised by an operation $\mathcal{F}_{\rho_{SA}}(N_{SA'})$ applied solely on A'.

Theorem 1. *Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$. Then, $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ in $[t_0, t_\infty)$, if and only if they satisfy*

$$\mathrm{Tr}_E P\left(|\Psi_{SEA}(t)\rangle\right) = \mathrm{Tr}_E P\left(|\tilde{\Psi}_{SEA}(t)\rangle\right) \tag{S7}$$

for all $t \in [t_0, t_\infty)$.

Thus, when the ME condition is satisfied, an identical time evolution on SA is sufficient to certify the equivalence, that is, we do not need to consider all possible sequences of quantum operations $\{\Gamma_i\}$. Towards the proof of the theorem, we show two lemmas.

Lemma 3. *Suppose a state $|\Psi_{SEA}\rangle$ on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, where $d_S = \dim\mathcal{H}_S = \dim\mathcal{H}_{A_1}$, and $|\Psi_{SEA}\rangle$ can be written as*

$$|\Psi_{SEA}\rangle = |\Upsilon_{SA_1}\rangle \otimes |\Phi_{EA_2}\rangle, \tag{S8}$$

where $|\Upsilon_{SA_1}\rangle$ is a maximally entangled state, and $\mathrm{Tr}_E P\left(|\Phi_{EA_2}\rangle\right)$ is a projector up to a proportionality constant. Then, for all finite-dimensional Hilbert spaces $\mathcal{H}_{A'}$ that include $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ as a subspace, and for all operators $N_{SA'}$ on $\mathcal{H}_S \otimes \mathcal{H}_{A'}$,

1. $\mathrm{supp}\mathrm{Tr}_{A'} P\left(N_{SA'}|\Psi_{SEA}\rangle\right) \subset \mathrm{supp}\mathrm{Tr}_A P\left(|\Psi_{SEA}\rangle\right)$
2. *For a given state $\rho_{SA} := \mathrm{Tr}_E P\left(|\Psi_{SEA}\rangle\right)$, there exists a linear map $\mathcal{F}_{\rho_{SA}}$ from $\mathfrak{B}\left(\mathcal{H}_S \otimes \mathcal{H}_{A'}\right)$ to $\mathfrak{B}\left(\mathcal{H}_{A'}\right)$ such that*

$$\mathcal{F}_{\rho_{SA}}\left(N_{SA'}\right) \otimes I_{SE}|\Psi_{SEA}\rangle = N_{SA'} \otimes I_E|\Psi_{SEA}\rangle, \tag{S9}$$

for all $N_{SA'} \in \mathfrak{B}\left(\mathcal{H}_S \otimes \mathcal{H}_{A'}\right)$.

Here, we note that the same $\mathcal{F}_{\rho_{SA}}$ satisfies Eq. (S9) for all states $|\Psi_{SEA}\rangle$ having the common reduced density matrix $\rho_{SA} = \mathrm{Tr}_E P\left(|\Psi_{SEA}\rangle\right)$. Figure 4 depicts the equivalence relation of Eq. (S9).

(Proof)

The proof of the first statement proceeds as follows. We

write $|\Upsilon_{SA_1}\rangle$ and $|\Phi_{EA_2}\rangle$ as

$$|\Upsilon_{SA_1}\rangle = \frac{1}{\sqrt{d_S}}\sum_{i=1}^{d_S}|e_i\rangle\otimes|i\rangle,$$
$$|\Phi_{EA_2}\rangle = \frac{1}{\sqrt{r}}\sum_{j=1}^{r}|f_j\rangle\otimes|j\rangle,$$

where $\{|e_i\rangle\}_{i=1}^{d_S}$, $\{|f_j\rangle\}_{j=1}^{d_E}$, $\{|i\rangle\}_{i=1}^{d_S}$, and $\{|j\rangle\}_{j=1}^{\dim\mathcal{H}_{A_2}}$ are orthonormal bases of $\mathcal{H}_S, \mathcal{H}_E, \mathcal{H}_{A_1}$, and $\mathcal{H}_{A_2}$, respectively, and r is the Schmidt rank [3] of $|\Phi_{EA_2}\rangle$. Without loss of generality, we see

$$N_{SA'}\otimes I_E|\Psi_{SEA}\rangle = \sum_{\alpha=1}^{R}\sqrt{\lambda_\alpha}|\xi_\alpha\rangle_{A'}\otimes|\eta_\alpha\rangle_{SE}, \quad \text{(S10)}$$

where $\{|\xi_\alpha\rangle\}_{\alpha=1}^{\dim A'}$ and $\{|\eta_\alpha\rangle\}_{\alpha=1}^{d_S d_E}$ are orthonormal bases of $\mathcal{H}_{A'}$ and $\mathcal{H}_S\otimes\mathcal{H}_E$, respectively. Also, R and $\{\lambda_\alpha\}_{\alpha=1}^{R}$ are the Schmidt rank and the Schmidt coefficients [3] of $N_{SA'}\otimes I_E|\Psi_{SEA}\rangle$. Naturally, equation $\mathrm{Tr}_{SA'}P(N_{SA'}|\Psi_{SEA}\rangle) = \mathrm{Tr}_{SA}P(|\Psi_{SEA}\rangle)$ holds, and by substituting the above expressions into each side we have

$$\mathrm{Tr}_S\sum_{\alpha}^{R}\lambda_\alpha|\eta_\alpha\rangle\langle\eta_\alpha|_{SE} = \frac{1}{r}\sum_{j=1}^{r}|f_j\rangle\langle f_j|, \quad \text{(S11)}$$

which means $\mathrm{supp}\mathrm{Tr}_S|\eta_\alpha\rangle\langle\eta_\alpha| \subset \mathrm{span}\{|f_j\rangle\}_{j=1}^{r}$ for all α. Then, $|\eta_\alpha\rangle$ should be in the space spanned by $\{|e_i\rangle|f_j\rangle\}_{i=1,j=1}^{d_S,\ r}$, which in turn implies $\mathrm{span}\{|\eta_\alpha\rangle\}_{\alpha=1}^{R}\subset\mathrm{span}\{|e_i\rangle|f_j\rangle\}_{i=1,j=1}^{d_S,\ r}$, hence statement 1.

We now move on to the proof the second statement. Suppose $\{|g_\alpha\rangle\}_{\alpha=1}^{d_{A'}}$ is an orthonormal basis of $\mathcal{H}_{A'}$, where $d_{A'} = \dim\mathcal{H}_{A'}$. There is a natural linear isomorphism, f, from $\mathcal{H}_{A'}\otimes\mathrm{supp}\mathrm{Tr}_A P(|\Psi_{SEA}\rangle)$ to $\mathfrak{B}(\mathrm{supp}\mathrm{Tr}_A P(|\Psi_{SEA}\rangle), \mathcal{H}_{A'})$ ($\mathfrak{B}(\mathcal{H},\mathcal{K})$ is a space of all linear operators from $\mathcal{H}$ to $\mathcal{K}$). f is defined by the correspondence of their bases:

$$f: |g_\alpha\rangle\otimes|e_i\rangle\otimes|f_j\rangle \mapsto |g_\alpha\rangle\langle e_i|\otimes\langle f_j| \quad (\forall\alpha,i,j). \quad \text{(S12)}$$

Then, we define a linear map $\mathcal{F}_{\rho_{SA}}$ as

$$\mathcal{F}_{\rho_{SA}}(N_{SA'}) := \sqrt{d_S r}\, f\left(N_{SA'}|\Psi_{SEA}\rangle\right)V, \quad \text{(S13)}$$

where $N_{SA'}\in\mathfrak{B}(\mathcal{H}_S\otimes\mathcal{H}_{A'})$, and V is a partial isometry defined as

$$V := \sum_{i=1}^{d_S}\sum_{j=1}^{r}|e_i\rangle_S|f_j\rangle_E\langle i|_{A_1}\langle j|_{A_2}.$$

We note that $\mathcal{F}_{\rho_{SA}}$ is well defined thanks to statement 1. Then, it is straightforward to see that $\mathcal{F}_{\rho_{SA}}$ satisfies Eq. (S9).

It is possible to show that the linear map $\mathcal{F}_{\rho_{SA}}$ only depends on a reduced density of matrix ρ_{SA} of the state $|\Psi\rangle$; more specifically, its effect does not depend on the Schmidt bases $\{|e_i\rangle\}_{i=1}^{d_S}$ and $\{|f_i\rangle\}_{i=1}^{d_E}$. Although f and V do depend on $\{|e_i\rangle\}_{i=1}^{d_S}$ and $\{|f_i\rangle\}_{i=1}^{d_E}$, their dependence cancels out in Eq. (S13). As a result, any specific choice of these bases does not affect the action of $\mathcal{F}_{\rho_{SA}}$. We can also confirm this property of $\mathcal{F}_{\rho_{SA}}$ from the fact that equation $\mathcal{F}_{\rho_{SA}}(N_{SA'})U_E|\Psi_{SEA}\rangle = N_{SA'}U_E|\Psi_{SEA}\rangle$ holds for all unitary operators U_E on $\mathcal{H}_E$. Therefore, statement 2 holds. ■

Lemma 4. *Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$. Then, for all instances $\{t_i\}_{i=1}^{n}$, where $t_0 < t_i < t_j < t_\infty$ for $i<j$, all finite dimensional Hilbert spaces $\mathcal{H}_{A'}\supset\mathcal{H}_A$, and all sets of linear operators $\left\{N_{SA'}^{(i)}\right\}_{i=1}^{n}$ on $\mathcal{H}_S\otimes\mathcal{H}_{A'}$, the following equation holds:*

$$\begin{aligned}&\prod_{j=1}^{n}\left(N_{SA'}^{(j)}U_{SE}^{(j)}\right)|\Psi_{SEA}\rangle\\ &=\left(\left(\mathcal{F}_{\rho_{SA}(t_n)}\cdot\prod_{j=2}^{n}N_{SA'}^{(j)}\mathcal{F}_{\rho_{SA}(t_{j-1})}\right)\left(N_{SA'}^{(1)}\right)\right)\otimes I_{SE}\\ &\qquad|\Psi_{SEA}(t_n)\rangle, \end{aligned}\quad \text{(S14)}$$

where $N_{SA'}\mathcal{F}_\rho$ is a linear map on $\mathfrak{B}(\mathcal{H}_S\otimes\mathcal{H}_{A'})$ defined as $N_{SA'}\mathcal{F}_\rho(M_{SA}) = N_{SA'}\left(\mathcal{F}_\rho(M_{SA'})\otimes I_S\right)$ for $M_{SA'}\in\mathfrak{B}(\mathcal{H}_S\otimes\mathcal{H}_{A'})$.

(Proof)
By repeatedly applying Lemma 3, we have:

$$\begin{aligned}&\prod_{j=1}^{n}\left(N_{SA'}^{(j)}U_{SE}^{(j)}\right)|\Psi_{SEA}\rangle\\ &=\prod_{j=2}^{n}\left(N_{SA'}^{(j)}U_{SE}^{(j)}\right)N_{SA'}^{(1)}|\Psi_{SEA}(t_1)\rangle\\ &=\prod_{j=2}^{n}\left(N_{SA'}^{(j)}U_{SE}^{(j)}\right)\mathcal{F}_{\rho_{SA}(t_1)}\left(N_{SA'}^{(1)}\right)|\Psi_{SEA}(t_1)\rangle\\ &=\prod_{j=3}^{n}\left(N_{SA'}^{(j)}U_{SE}^{(j)}\right)N_{SA'}^{(2)}\mathcal{F}_{\rho_{SA}(t_1)}\left(N_{SA'}^{(1)}\right)|\Psi_{SEA}(t_2)\rangle\\ &=\prod_{j=3}^{n}\left(N_{SA'}^{(j)}U_{SE}^{(j)}\right)\mathcal{F}_{\rho_{SA}(t_1)}\left(N_{SA'}^{(2)}\mathcal{F}_{\rho_{SA}(t_1)}\left(N_{SA'}^{(1)}\right)\right)\\ &\qquad|\Psi_{SEA}(t_2)\rangle\\ &=\left(\left(\mathcal{F}_{\rho_{SA}(t_n)}\cdot\prod_{j=2}^{n}N_{SA'}^{(j)}\mathcal{F}_{\rho_{SA}(t_{j-1})}\right)\left(N_{SA'}^{(1)}\right)\right)\otimes I_{SE}\\ &\qquad|\Psi_{SEA}(t_n)\rangle.\end{aligned}$$

Thus, we derived Eq. (S14) ■

We are now ready to prove Theorem 1.

(Proof of Theorem 1)
The "*only if*" part is trivial from the definition. We therefore prove the "*if*" part of the statement. Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition and

$(d_E, |\Psi_{SEA}\rangle, H_{SE})$ and $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ satisfy Eq. (S7). Then, for all instances $\{t_i\}_{i=1}^n$, where $t_0 < t_i < t_j < t_\infty$ for $i < j$, all finite-dimensional Hilbert spaces $\mathcal{H}_{A'} \supset \mathcal{H}_A$ and all sets of linear operators $\left\{N_{SA'}^{(i)}\right\}_{i=1}^n$ on $\mathcal{H}_S \otimes \mathcal{H}_{A'}$, we derive the following equations:

$$\begin{aligned}
&\mathrm{Tr}_E P\left(\prod_{j=1}^n \left(N_{SA'}^{(j)} U_{SE}^{(j)}\right) |\Psi_{SEA}\rangle\right) \\
=&\left(\left(\mathcal{F}_{\rho_{SA'}(t_n)} \cdot \prod_{j=2}^n N_{SA'}^{(j)} \mathcal{F}_{\rho_{SA}(t_{j-1})}\right)\left(N_{SA'}^{(1)}\right)\right) \otimes I_S \cdot \\
&\quad \mathrm{Tr}_E P\Big(|\Psi_{SEA}(t_n)\rangle\Big) \cdot \\
&\left(\left(\mathcal{F}_{\rho_{SA}(t_n)} \cdot \prod_{j=2}^n N_{SA'}^{(j)} \mathcal{F}_{\rho_{SA}(t_{j-1})}\right)\left(N_{SA'}^{(1)}\right)\right)^\dagger \otimes I_S \\
=&\left(\left(\mathcal{F}_{\rho_{SA}(t_n)} \cdot \prod_{j=2}^n N_{SA'}^{(j)} \mathcal{F}_{\rho_{SA}(t_{j-1})}\right)\left(N_{SA'}^{(1)}\right)\right) \otimes I_S \cdot \\
&\quad \mathrm{Tr}_E P\Big(|\tilde{\Psi}_{SEA}(t_n)\rangle\Big) \cdot \\
&\left(\left(\mathcal{F}_{\rho_{SA}(t_n)} \cdot \prod_{j=2}^n N_{SA'}^{(j)} \mathcal{F}_{\rho_{SA}(t_{j-1})}\right)\left(N_{SA'}^{(1)}\right)\right)^\dagger \otimes I_S \\
=&\mathrm{Tr}_E P\left(\prod_{j=1}^n \left(N_{SA'}^{(j)} \tilde{U}_{SE}^{(j)}\right) |\tilde{\Psi}_{SEA}\rangle\right),
\end{aligned}$$

where Eq. (S14) of Lemma 4 is used in the first and third equalities. The above equation is nothing but Eq. (S2) in Lemma 1, and thus the "*if*" part of the theorem holds. ■

Theorem 1 leads to the following corollary.

Corollary 1. *Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$, and $d_E < +\infty$. Then, $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ in $[t_0, t_\infty)$, and $\tilde{d}_E < +\infty$ imply that $(d_E, |\Psi_{SEA}(\tilde{t}_0)\rangle, H_{SE}) \equiv (\tilde{d}_E, |\tilde{\Psi}_{SEA}(t_0)\rangle, \tilde{H}_{SE})$ in any time interval $[\tilde{t}_0, \tilde{t}_\infty)$.*

Roughly speaking, what this claims is as follows. Suppose there are two triples, the one with E and the other with $\tilde{E}$. If the ME condition is fulfilled by one of them and both give rise to the identical natural time evolution on SA for a finite duration, regardless of its length, then both triples are equivalent; that is, any active quantum operations on SA cannot reveal the difference between them.

(Proof)
Since both d_E and $\tilde{d}_E$ are finite, both $\mathrm{Tr}_E P(|\Psi_{SEA}(t)\rangle)$ and $\mathrm{Tr}_E P\left(|\tilde{\Psi}_{SEA}(t)\rangle\right)$ are analytical functions and coincide on $[t_0, t_\infty)$. Thus, by analytical continuation, $\mathrm{Tr}_E P(|\Psi_{SEA}(z)\rangle) = \mathrm{Tr}_E P\left(|\tilde{\Psi}_{SEA}(z)\rangle\right)$ holds for all complex numbers z. Together with Theorem 1, we reach the statement of the corollary. ■

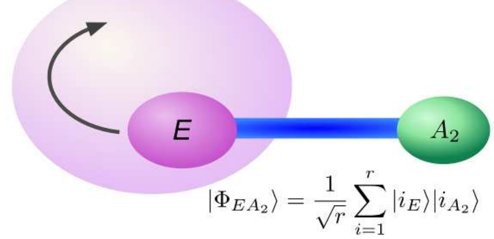

FIG. 5: A situation in which the ME condition may be satisfied, but the support of the state $|\Phi_{EA_2}\rangle$ in E moves around in a larger space. Theorem 2 rules out such a structure once the ME condition is fulfilled.

B. The ME condition implies maximal entanglement between A and SE

In the previous subsection, we have suggested that when the triple $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$, there exists an equivalent triple in which $|\Phi_{EA_2(t)}\rangle$ can be chosen as a maximally entangled state on the entire space of $\mathcal{H}_E \otimes \mathcal{H}_A$. Now we shall prove it as a theorem in this subsection.

Theorem 2. *Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$, and $d_E < +\infty$. Then, there exists a triple $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ such that $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE}) \equiv (d_E, |\Psi_{SEA}\rangle, H_{SE})$ in $[t_0, t_\infty)$, and $|\tilde{\Psi}_{SEA}\rangle$ is a maximally entangled state on the full space of SEA with respect to the partition between SE and A; that is, $\mathrm{Tr}_A P\left(|\tilde{\Psi}_{SEA}\rangle\right) = \frac{1}{d_S d_E} I_S \otimes I_E$.*

Let us sketch the idea of its proof: For $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfying the ME condition, $|\Psi_{SEA}\rangle$ is a maximally entangled state on the full space of S, E, and A if and only if $d_E = \mathrm{rank}\rho_E(t)$, where $\rho_E(t) := \mathrm{Tr}_{SA} P\left(e^{-iH_{SE}(t-t_0)}|\Psi_{SEA}\rangle\right)$. If the rank of $\rho_E(t)$ is smaller than d_E, $\mathrm{supp}\rho_E(t)$ may move around within E as time proceeds (see Fig. 5 for an intuitive illustration). In such a case, the property of maximally entangled pairs, Eq. (7) in the main text, cannot be used for our Hamiltonian identification purpose. Thus, in order to prove the theorem, we need to show the existence of Hamiltonian H'_{SE} such that $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (d_E, |\Psi_{SEA}\rangle, H'_{SE})$ and $[H'_{SE}, \rho_{SE}(t_0)]=0$. With such a H'_{SE}, $\rho'_E(t) := \mathrm{Tr}_{SA} P\left(e^{-iH'_{SE}(t-t_0)}|\Psi_{SEA}\rangle\right)$ will be time-independent.

To this end, we restrict the space $\mathcal{H}_E$ to $\mathrm{supp}\rho'_E(t)$, or equivalently, we define $\tilde{d}_E$, $|\tilde{\Psi}_{SEA}\rangle$, and $\tilde{H}_{SE}$ as $\tilde{d}_E := \mathrm{rank}\rho_E(t_0)$, $|\tilde{\Psi}_{SEA}\rangle := |\Psi_{SEA}\rangle$, and $\tilde{H}_{SE} := (I_{SA} \otimes$

$P_E(t_0))H'_{SE}(I_{SA}\otimes P_E(t_0))$, respectively, where $P_E(t) := \mathrm{rank}\rho'_E(t)\cdot\rho'_E(t)$ is a projector onto the support of ρ_E. Then, the triple $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE})$ satisfies the desired conditions: $(\tilde{d}_E, |\tilde{\Psi}_{SEA}\rangle, \tilde{H}_{SE}) \equiv (d_E, |\Psi_{SEA}\rangle, H_{SE})$ in $[t_0, t_\infty)$, and $|\tilde{\Psi}_{SEA}\rangle$ is a maximally entangled state on the full space of SEA. Therefore, the proof of the theorem can be reduced to constructing Hamiltonian H'_{SE} so that $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv (d_E, |\Psi_{SEA}\rangle, H'_{SE})$ and $[H'_{SE}, \rho_{SE}(t_0)]$=0. In fact, we shall see that the ME condition enables us to do so.

In order to construct the H'_{SE}, we need more detailed description of H_{SE}. In general, a Hamiltonian H_{SE} can be decomposed as

$$H_{SE} = I_S\otimes H_{\mathrm{id}} + \sum_{\alpha=1}^{d_S^2-1}\sigma_\alpha\otimes H_\alpha, \tag{S15}$$

where $\{I_S\}\bigcup\{\sigma_\alpha\}_{\alpha=1}^{d_S^2-1}$ forms an orthogonal basis of a real space of all Hermitian operators on $\mathcal{H}_S$. When $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition, the state on SE is proportional to $I_S\otimes P_E(t)$, where $P_E(t)$ is a projector-valued function on E. Then, the Schroedinger equation for $|\Psi_{SEA}(t)\rangle$ becomes

$$i\frac{d}{dt}I_S\otimes P_E(t) \tag{S16}$$
$$=[H_{SE}, I_S\otimes P_E(t)]$$
$$=I_S\otimes[H_{\mathrm{id}}, P_E(t)] + \sum_{\alpha=1}^{d_S^2-1}\sigma_\alpha\otimes[H_\alpha, P_E(t)], \tag{S17}$$

for all $t\in\mathbb{R}$. Thus, H_{id} and H_α satisfy

$$i\frac{d}{dt}P_E(t) = [H_{\mathrm{id}}, P_E(t)], \tag{S18}$$
$$0 = [H_\alpha, P_E(t)] \quad (\forall\alpha). \tag{S19}$$

That is, only H_{id} is responsible for the time evolution of $\rho_E(t) = P_E(t)/\mathrm{rank}\rho_E(t_0)$. Therefore, a modification to H_{id} may lead to desired Hamiltonian H'_{SE}. Suppose that the desired modification is described as $H_{\mathrm{id}}\to H_{\mathrm{id}} - H'_{\mathrm{id}}$ with a Hermitian operator H'_{id}. Such a H'_{id} is shown to exist and has properties as in the following lemma.

Lemma 5. *Suppose $\mathcal{H}_E$ is a finite-dimensional Hilbert space, and suppose a projector-valued function $P_E(t)$ on $\mathcal{H}_E$, and Hamiltonians H_{id} and $\{H_\alpha\}_{\alpha=1}^c$ on $\mathcal{H}_E$ satisfies Eqs. (S18) and (S19) for all $t\in I_t$, where I_t is an open subset of $\mathbb{R}$. Then, there exists Hamiltonian H'_{id} that satisfies the following equations:*

$$[H_{\mathrm{id}}, P_E(t)] = [H'_{\mathrm{id}}, P_E(t)], \quad (t\in I_t) \tag{S20}$$
$$[H_{\mathrm{id}}, H'_{\mathrm{id}}] = 0, \tag{S21}$$
$$[H_\alpha, H'_{\mathrm{id}}] = 0 \;\; (\forall\alpha). \tag{S22}$$

(Proof)
We generate a sequence of triples $\left\{\left(\mathfrak{g}_n, G_n, H_{\mathrm{id}}^{(n)}\right)\right\}_{n=1}^\infty$, where $H_{\mathrm{id}}^{(n)}$ are Hamiltonians, $\mathfrak{g}_n$ are linear Lie algebras on $\mathcal{H}_E$, and G_n are Lie groups [13], according to the following procedure. Suppose H_α have spectral decompositions as $H_\alpha = \sum_j h_j^{(\alpha)}P_j^{(\alpha)}$, where $h_j^{(\alpha)}\neq h_{j'}^{(\alpha)}$ for all $j\neq j'$, and $P_j^{(\alpha)}$ are all projectors. Then, $\mathfrak{g}_1$ is defined as a Lie algebra whose elements are $\left\{iP_j^{(\alpha)}\right\}$ $(\forall j,\alpha)$, G_1 is a compact Lie group generated by $\mathfrak{g}_1$, and $H_{\mathrm{id}}^{(1)}$ is defined as

$$H_{\mathrm{id}}^{(1)} := \int_{U\in G_1} UH_{\mathrm{id}}U^\dagger d\mu(U), \tag{S23}$$

where $\mu(U)$ is a Haar measure [13, 14] on G_1. Starting from $\left(\mathfrak{g}_1, G_1, H_{\mathrm{id}}^{(1)}\right)$, we recurrently define $\left(\mathfrak{g}_n, G_n, H_{\mathrm{id}}^{(n)}\right)$ as follows: Suppose $H_{\mathrm{id}} - H_{\mathrm{id}}^{(n)}$ has a spectral decomposition as $H_{\mathrm{id}} - H_{\mathrm{id}}^{(n)} = \sum_j h_j Q_j^{(n)}$, where $Q_j^{(n)}$ are projectors. Then, we let $\mathfrak{g}_{n+1}$ and G_{n+1} be a linear Lie algebra consisting of $\mathfrak{g}_n\bigcup\left\{iQ_j^{(n)}\right\}$ $(\forall j)$ and a compact Lie group generated by $\mathfrak{g}_{n+1}$. Similarly as above, $H_{\mathrm{id}}^{(n+1)}$ is defined as

$$H_{\mathrm{id}}^{(n+1)} := \int_{U\in G_{n+1}} UH_{\mathrm{id}}^{(n)}U^\dagger d\mu(U). \tag{S24}$$

Then, we can prove the following equations for all $t\in I_t$ (The time t will denote $t\in I_t$ hereafter unless otherwise stated):

$$[H_\alpha, H_{\mathrm{id}}^{(n)}] = 0, \quad (\forall\alpha, n) \tag{S25}$$
$$[H_0 - H_{\mathrm{id}}^{(n)}, H_{\mathrm{id}}^{(n+1)}] = 0, \quad (\forall n) \tag{S26}$$
$$[H_{\mathrm{id}}, P_E(t)] = [H_{\mathrm{id}}^{(n)}, P_E(t)], \quad (\forall n). \tag{S27}$$

Note that Eq. (S37) can be reexpressed as $[Q_j^{(n)}, P_E(t)] = 0$ by subtracting the RHS from the LHS.

Let us first show the following two equations:

$$\left[P_j^{(\alpha)}, H_{\mathrm{id}}^{(n)}\right] = 0, \quad (\forall j,\alpha,n) \tag{S28}$$

and

$$\left[Q_j^{(n)}, H_{\mathrm{id}}^{(n+1)}\right] = 0 \quad (\forall j,n), \tag{S29}$$

from which Eqs. (S25) and (S26) are directly obtained. Since $U_\delta := I_E - P_j^{(\alpha)} + e^{i\delta}P_j^{(\alpha)}$ is in G_n for all $\delta\in[0,2\pi]$, $H_{\mathrm{id}}^{(n)}$ satisfies

$$\begin{aligned}&H_{\mathrm{id}}^{(n)}\\ =&\int_{\delta\in[0,2\pi]}\left(I - P_j^{(\alpha)} + e^{i\delta}P_j^{(\alpha)}\right)H_{\mathrm{id}}^{(n)}\\ &\cdot\left(I - P_j^{(\alpha)} + e^{-i\delta}P_j^{(\alpha)}\right)\frac{d\delta}{2\pi}\\ =&(I-P_j^{(\alpha)})H_{\mathrm{id}}^{(n)}(I-P_j^{(\alpha)}) + P_j^{(\alpha)}H_{\mathrm{id}}^{(n)}P_j^{(\alpha)},\end{aligned}$$

which implies

$$P_j^{(\alpha)} H_{\rm id}^{(n)} = P_j^{(\alpha)} H_{\rm id}^{(n)} P_j^{(\alpha)} = H_{\rm id}^{(n)} P_j^{(\alpha)}.$$

Thus, Eq. (S28) holds. Similarly, we can prove Eq. (S29) from the fact $U_\delta := I_E - Q_j^{(n)} + e^{i\delta} Q_j^{(n)}$ in G_{n+1}.

We now prove Eq. (S27) by induction. First, we consider the case $n = 1$. Equation (S19) implies that $P_j^{(\alpha)}$ satisfies $\left[P_j^{(\alpha)}, P_E(t)\right] = 0$. Then, by differentiating this equation and using Eq. (S18), we obtain

$$\left[P_j^{(\alpha)}, [H_{\rm id}, P_E(t)]\right] = 0 \tag{S30}$$

for all j and α. Then, because of the following identities,

$$\begin{aligned}
&\left[P_j^{(\alpha)}, P_E(t)\right] = 0,\\
&\left[P_j^{(\alpha)}, [H_{\rm id}, P_E(t)]\right] = 0,\\
&\left[\left[P_{i_1}^{(\alpha_1)}, P_{i_2}^{(\alpha_2)}\right], P_{i_3}^{(\alpha_3)}\right] + \left[\left[P_{i_2}^{(\alpha_2)}, P_{i_3}^{(\alpha_3)}\right], P_{i_1}^{(\alpha_1)}\right]\\
&+ \left[\left[P_{i_3}^{(\alpha_3)}, P_{i_2}^{(\alpha_2)}\right], P_{i_1}^{(\alpha_1)}\right] = 0,
\end{aligned}$$

the last of which is the Jacobi identify, all operators written in the form of

$$\left[\cdots\left[\left[P_{i_1}^{(\alpha_1)}, P_{i_2}^{(\alpha_2)}\right], P_{i_3}^{(\alpha_3)}\right]\cdots, P_{i_k}^{(\alpha_k)}\right]. \tag{S31}$$

commute with $P_E(t)$ and $[H_{\rm id}, P_E(t)]$. Thus, since an arbitrary $X \in \mathfrak{g}_1$ can be written as a linear combination of the terms in the form of Eq. (S31), any $X \in \mathfrak{g}_1$ satisfies $[X, P_E(t)] = 0$ and $[X, [H_{\rm id}, P_E(t)]] = 0$.

Since G_1 is a compact Lie group, for a given $U \in G_1$, there exists an operator $X \in \mathfrak{g}_1$ satisfying $U = \exp X$. Thus, an arbitrary $U \in G_1$ satisfies

$$[U, P_E(t)] = 0 \tag{S32}$$

and

$$[U, [H_{\rm id}, P_E(t)]] = 0. \tag{S33}$$

Hence, by using Eqs. (S32) and (S33), we can derive the following equation for all $U \in G_1$:

$$\begin{aligned}
[H_{\rm id}, P_E(t)] &= U\,[H_{\rm id}, P_E(t)]\,U^\dagger\\
&= \left[U H_{\rm id} U^\dagger, P_E(t)\right].
\end{aligned}$$

By integrating the above equation on G_1 with a Haar measure $\mu(U)$, we obtain $[H_{\rm id}, P_E(t)] = \left[H_{\rm id}^{(1)}, P_E(t)\right]$.

Second, following a discussion similar to the case of $n = 1$, we prove Eq. (S27) for $n+1$ assuming that Eq. (S27) holds for all $k \le n$, i.e., $\left[Q_j^{(k)}, P_E(t)\right] = 0$ for all j, $k \le n$. As in the case of $P_j^{(\alpha)}$, this fact implies

$$\left[Q_j^{(k)}, [H_{\rm id}, P_E(t)]\right] = 0 \tag{S34}$$

for all j, $k \le n$. Then, since $\left\{iQ_j^{(k)}\right\}_j$ with all $k \le n$ and $\left\{iP_j^{(\alpha)}\right\}_{j,\alpha}$ generate G_{n+1}, Eqs. (S30) and (S34) imply that an arbitrary $X \in \mathfrak{g}_{n+1}$ satisfies $[X, P_E(t)] = 0$ and $[X, [H_{\rm id}, P_E(t)]] = 0$. Therefore, as in the case of $n = 1$, we deduce $[H_{\rm id}, P_E(t)] = \left[H_{\rm id}^{(n+1)}, P_E(t)\right]$.

Equations. (S25), (S26), and (S27) can be used in the final step to prove the lemma. Since all $\mathfrak{g}_n$ are Lie sub-algebras of $\mathfrak{su}(\dim \mathcal{H}_E)$, and thus finite-dimensional, there exists an $N \in \mathbb{N}$ such that $\mathfrak{g}_n = \mathfrak{g}_{n+1}$ for all $n \ge N$ (note $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$). Therefore, we have

$$[H_\alpha, H_{\rm id}^{(N)}] = 0, \tag{S35}$$

$$[H_{\rm id} - H_{\rm id}^{(N)}, H_{\rm id}^{(N)}] = 0, \tag{S36}$$

$$[H_{\rm id}, P_E(t)] = [H_{\rm id}^{(N)}, P_E(t)]. \tag{S37}$$

Taking $H'_{\rm id} := H_{\rm id}^{(N)}$ proves the lemma. ■

Let us proceed to the proof of Theorem 2.

(Proof of Theorem 2)

Suppose $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$, i.e., $P_E(t) := r{\rm Tr}_{SA} P\left(|\Psi_{SEA}(t)\rangle\right)$ is a projector-valued function, where $r := \dim {\rm supp}{\rm Tr}_{SA} P\left(|\Psi_{SEA}\rangle\right)$. Therefore, Eqs. (S18) and (S19) hold with $H_{\rm id}$ and $\{H_\alpha\}_{\alpha=1}^c$ defined in Eq. (S15). Lemma 5 then tells that there exists a Hermitian operator $H'_{\rm id}$ on $\mathcal{H}_E$ that satisfies Eqs. (S20)-(S22) for all $I_t \subset \mathbb{R}$. We now define H'_{SE} as

$$H'_{SE} := H_{SE} - I_S \otimes H'_{\rm id}. \tag{S38}$$

Then, $[H'_{SE}, I_S \otimes H'_{\rm id}]$ and $[H'_{SE}, I_S \otimes P_E(t)]$ can be computed as

$$\begin{aligned}
[H'_{SE}, I_S \otimes H'_{\rm id}] =& I_S \otimes [H_{\rm id} - H'_{\rm id}, H'_{\rm id}]\\
=&0
\end{aligned} \tag{S39}$$

and

$$\begin{aligned}
[H'_{SE}, I_S \otimes P_E(t)] &= I_S \otimes [H_{\rm id} - H'_{\rm id}, P_E(t)]\\
&= 0,
\end{aligned} \tag{S40}$$

where we have used Eqs. (S21) and (S22) in Eq.(S39), and Eq. (S20) in Eq. (S40). Then, by defining $|\Psi'_{SEA}(t)\rangle := \exp\left(-iH'_{SE}(t - t_0)\right)|\Psi_{SEA}\rangle$, we obtain

$$\begin{aligned}
&{\rm Tr}_E P\left(|\Psi_{SEA}(t)\rangle\right)\\
=&{\rm Tr}_E \exp\left(-iH_{SE}(t-t_0)\right) P\left(|\Psi_{SEA}\rangle\right) \exp\left(iH_{SE}(t-t_0)\right)\\
=&{\rm Tr}_E \exp\left(-i\left(I_S \otimes H'_{\rm id}\right)(t-t_0)\right) \exp\left(-iH'_{SE}(t-t_0)\right)\\
&\quad \cdot P\left(|\Psi_{SEA}\rangle\right) \exp\left(iH_{SE}(t-t_0)\right)\\
&\quad \cdot \exp\left(i\left(I_S \otimes H'_{\rm id}\right)(t-t_0)\right)\\
=&{\rm Tr}_E \exp\left(-iH'_{SE}(t-t_0)\right) P\left(|\Psi_{SEA}\rangle\right) \exp\left(iH'_{SE}(t-t_0)\right)\\
=&{\rm Tr}_E P\left(|\Psi'_{SEA}(t)\rangle\right),
\end{aligned} \tag{S41}$$

where we have used Eqs. (S38) and (S39) in the second and third equalities, respectively. Also, since the state

on SE can be written as

$$\begin{aligned}&Tr_A P\left(|\Psi'_{SEA}(t)\rangle\right)\\ =&\frac{1}{rd_S}\exp\left(-iH'_{SE}(t-t_0)\right)\\ &\cdot\left(I_S\otimes P_E(t_0)\right)\exp\left(iH'_{SE}(t-t_0)\right)\\ =&\frac{1}{rd_S}I_S\otimes P_E(t_0),\end{aligned}\tag{S42}$$

the state $\mathrm{Tr}_{SA}P\left(|\Psi'_{SEA}(t)\rangle\right)=\frac{1}{r}P_E(t_0)$ does not depend on time t. Therefore, by defining a Hilbert space $\tilde{\mathcal{H}}_E$, a state $|\tilde{\Psi}_{SEA}\rangle$, and a Hamiltonian $\tilde{H}_{SE}$ on $\mathcal{H}_S\otimes\tilde{\mathcal{H}}_E$ as

$$\tilde{\mathcal{H}}_E := \mathrm{supp}P_E(t_0),\tag{S43}$$

$$|\tilde{\Psi}_{SEA}\rangle := |\Psi'_{SEA}(t_0)\rangle = |\Psi_{SEA}\rangle,\tag{S44}$$

$$\tilde{H}_{SE} := I_S\otimes P_E(t_0)H'_{SE}I_S\otimes P_E(t_0),\tag{S45}$$

the triple $\left(\tilde{d}_E,|\tilde{\Psi}_{SEA}\rangle,\tilde{H}_{SE}\right)$ satisfies the statement of the theorem, where $\tilde{d}_E:=\dim\tilde{\mathcal{H}}_E=\mathrm{rank}P_E(t_0)$. ■

An important corollary of Theorem 2 follows:

Corollary 2. *Suppose $(d_E,|\Psi_{SEA}\rangle,H_{SE})$ satisfies the ME condition in $[t_0,t_\infty)$, and $\dim d_E<+\infty$. Then, $(d_E,|\Psi_{SEA}(t'_0)\rangle,H_{SE})$ with $|\Psi_{SEA}(t'_0)\rangle$ being given by Eq. (S4) also satisfies the ME condition in $[t'_0,t'_\infty)$ for all t'_0 and t'_∞ $(t'_0<t'_\infty)$.*

(Proof)

Theorem 2 guarantees that there exists $(\tilde{d}_E,|\tilde{\Psi}_{SEA}\rangle,\tilde{H}_{SE})$ such that $(d_E,|\Psi_{SEA}\rangle,H_{SE})\equiv(\tilde{d}_E,|\tilde{\Psi}_{SEA}\rangle,\tilde{H}_{SE})$ in $[t_0,t_\infty)$ and $|\tilde{\Psi}_{SEA}\rangle$ is a MES with respect to the partition $A|SE$. Then, for all t'_0 and t'_∞ satisfying $t'_0<t'_\infty$, $(d_E,|\Psi_{SEA}(t'_0)\rangle,H_{SE})\equiv(\tilde{d}_E,|\tilde{\Psi}_{SEA}(t'_0)\rangle,\tilde{H}_{SE})$ in $[t'_0,t'_\infty)$, as stated in Corollary 1. On the other hand, since $|\tilde{\Psi}_{SEA}\rangle$ is a MES, $(\tilde{d}_E,|\tilde{\Psi}_{SEA}(t'_0)\rangle,\tilde{H}_{SE})$ satisfies the ME condition in any time intervals $[t'_0,t'_\infty)$ $(t'_0<t'_\infty)$. Recalling the definition of the triple equivalence (and also the "if" part of Theorem 1), this leads to the statement of the corollary. ■

This corollary tells that if our system satisfies the ME condition for a non-zero time period, no matter how short it is, it will always satisfy the ME condition from then on.

III. TOMOGRAPHY OF THE ENVIRONMENT UNDER THE ME CONDITION

In this section, we show that, when the entire system SEA satisfies the ME condition, it is possible to specify the equivalence class by performing tomography on joint system SA. More precisely, when our triple $(d_E,|\Psi_{SEA}\rangle,H_{SE})$ satisfies the ME condition in the time period $[t_0,t_\infty)$, we can determine $\tilde{d}_E$, $|\tilde{\Psi}_{SEA}\rangle$ and $\tilde{H}_{SE}$ that satisfy $(d_E,|\Psi_{SEA}\rangle,H_{SE})\equiv\left(\tilde{d}_E,|\tilde{\Psi}_{SEA}\rangle,\tilde{H}_{SE}\right)$ in $[t_0,t_\infty)$. As we have explained in Sec. I, this information is sufficient to control system E and exploit it as a resource for quantum computation.

Let us start with another corollary of Theorem 2:

Corollary 3. *Suppose a triple $(d_E,|\Psi_{SEA}\rangle,H_{SE})$ satisfies the ME condition in $[t_0,t_\infty)$. Then, there exist Hilbert spaces $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$ and a triple $\left(\tilde{d}_E,|\tilde{\Psi}_{SEA}\rangle,\tilde{H}_{SE}\right)$, such that $\mathcal{H}_A=\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}$, and $|\tilde{\Psi}_{SEA}\rangle$ can be written as*

$$|\tilde{\Psi}_{SEA}\rangle=|\Upsilon_{SA_1}\rangle\otimes|\Upsilon_{EA_2}\rangle,\tag{S46}$$

where $|\Upsilon_{SA_1}\rangle\in\mathcal{H}_S\otimes\mathcal{H}_{A_1}$ and $|\Upsilon_{EA_2}\rangle\in\mathcal{H}_E\otimes\mathcal{H}_{A_2}$ are maximally entangled states, and $\tilde{d}_E=\dim\mathrm{supp}\mathrm{Tr}_{A_2}P\left(|\Upsilon_{EA_2}\rangle\right)$. Moreover, we can choose a basis set freely for both A_2 and E such that $|\Upsilon_{EA_2}\rangle$ can be expressed as

$$|\Upsilon_{EA_2}\rangle=\frac{1}{\sqrt{\tilde{d}_E}}\sum_{i=1}^{\tilde{d}_E}|i_{A_2}\rangle|i_E\rangle.\tag{S47}$$

The last part of the corollary states that there is freedom in the choice of a basis for A_2 and E due to the equivalence of observable dynamics induced by Hamiltonians that may be equivalent up to a local unitary U_E.

(Proof)

The first part of the statement is simply a rephrase of Theorem 2. Thus, we only prove the second part, which says that $|\Upsilon_{EA_2}\rangle$ can always be written in the form of Eq. (S47) even if we choose arbitrary bases. Suppose that $(d_E,|\Psi_{SEA}\rangle,H_{SE})$ satisfies the ME condition in $[t_0,t_\infty)$ and $|\tilde{\Psi}_{SEA}\rangle$ and $\tilde{H}_{SE}$ are given as $|\tilde{\Psi}_{SEA}\rangle:=(I\otimes U_E)|\Psi_{SEA}\rangle$ and $\tilde{H}_{SE}:=(I_S\otimes U_E)H_{SE}(I_S\otimes U_E^\dagger)$, respectively, where U_E is an arbitrary unitary operator on $\mathcal{H}_E$. Then, $|\Psi_{SEA}(t)\rangle:=e^{-iH_{SE}t}|\Psi_{SEA}\rangle$ and $|\tilde{\Psi}_{SEA}(t)\rangle:=e^{-i\tilde{H}_{SE}t}|\tilde{\Psi}_{SEA}\rangle$ satisfy $\mathrm{Tr}_E P\left(|\Psi_{SEA}(t)\rangle\right)=\mathrm{Tr}_E\left(|\tilde{\Psi}_{SEA}(t)\rangle\right)$ $(\forall t\in[t_0,t_\infty)$, i.e., $(d_E,|\Psi_{SEA}\rangle,H_{SE})\equiv\left(\tilde{d}_E,|\tilde{\Psi}_{SEA}\rangle,\tilde{H}_{SE}\right)$. Hence, the state $|\Psi_{SEA}\rangle$ has a unitary freedom on E, i.e., the freedom in choosing a basis to express it as Eq. (S47). ■

Therefore, when our system satisfies the ME condition, we can determine the decomposition $\mathcal{H}_A=\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}$, the state $|\Upsilon_{SA_1}\rangle$, and the dimension $\tilde{d}_E$ satisfying $\tilde{d}_E=\dim\mathrm{supp}\mathrm{Tr}_{SEA_1}P\left(|\Psi_{SEA}\rangle\right)$ by performing (joint) state tomography on SA. Moreover, by redefining $\mathcal{H}_A:=\mathrm{supp}\mathrm{Tr}_{SE}P\left(|\Psi_{SEA}\rangle\right)$, we can assume $\dim\mathcal{H}_{A_2}=\dim\mathcal{H}_E=\tilde{d}_E$ and $\dim\mathcal{H}_A=d_S\tilde{d}_E$. The above corollary also implies that we can always assume system EA_2 is in the $\tilde{d}_E\times\tilde{d}_E$-dimensional standard maximally entangled state (of the form of Eq. (S47)).

The remaining task is now to determine the interaction Hamiltonian H_{SE} (finally!). Theorems 1 and 2 allow us to state the following:

Corollary 4. *Suppose a triple $(d_E, |\Psi_{SEA}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$. If a Hermitian matrix $\tilde{H}_{SE}$ on $\mathcal{H}_S \otimes \mathcal{H}_E$ satisfies*

$$i\frac{d}{dt}\rho_{SA}(t) = \left[I_S \otimes \tilde{H}_{SE}^T, \rho_{SA}(t)\right] \tag{S48}$$

for all t in a neighbourhood of t_0, then $(d_E, |\Psi_{SEA}\rangle, H_{SE}) \equiv \left(d_E, |\Psi_{SEA}\rangle, \tilde{H}_{SE}\right)$ in $[t_0, t_\infty)$. In Eq. (S48), $\rho_{SA}(t) := \mathrm{Tr}_E P\left(e^{-iH_{SE}(t-t_0)}|\Psi_{SEA}\rangle\right)$, the transposition T is taken with respect to the Schmidt basis of $|\Psi_{SEA}\rangle$ and $\tilde{H}_{SE}^T$ is an operator on $\mathcal{H}_A$.

Thanks to Corollary 3, we can take it for granted that $|\Psi_{SEA}\rangle$ has the form of Eq. (S46). Corollary 4 implies that, for a given $\rho_{SA}(t)$, which can be specified by state tomography, all Hermitian matrices $\tilde{H}_{SE}$ satisfying Eq. (S48) can be adopted as the interaction Hamiltonian between systems S and E.

Let us now describe how to find such a matrix $\tilde{H}_{SE}$ from the observed data of $\rho_{SA}(t)$. Since the time evolution of $\rho_{SA}(t)$ is induced by the (finite-dimensional) matrix $\tilde{H}_{SE}$, there exist a set of real numbers $\{\theta_\alpha\}_{\alpha=1}^L$ and a set of linear operators $\{\rho_\alpha\}_{\alpha=0}^L$ on $\mathcal{H}_S \otimes \mathcal{H}_A$ such that $\rho_{SA}(t)$ can be written as

$$\rho_{SA}(t) = \rho_0 + \sum_{\alpha=1}^{L}\left(e^{i\theta_\alpha(t-t_0)}\rho_\alpha + e^{-i\theta_\alpha(t-t_0)}\rho_\alpha^\dagger\right), \tag{S49}$$

where L is at most $d_S\tilde{d}_E\left(d_S\tilde{d}_E - 1\right)/2$ and ρ_0 is Hermitian. Setting $t = t_0$, we have

$$\rho_{SA}(t_0) = \rho_0 + \sum_{\alpha=1}^{L}\left(\rho_\alpha + \rho_\alpha^\dagger\right). \tag{S50}$$

Further, differentiating Eq. (S49) n times leads to

$$\frac{d^n}{dt^n}\rho_{SA}(t)\Big|_{t=t_0} = \sum_{\alpha=1}^{L}\left\{(i\theta_\alpha)^n\rho_\alpha + (-i\theta_\alpha)^n\rho_\alpha^\dagger\right\}. \tag{S51}$$

Therefore, we can determine $\{\theta_\alpha\}_{\alpha=1}^L$ and $\{\rho_\alpha\}_{\alpha=0}^L$ from at most d_A^2-th order derivative of $\rho_{SA}(t)$ at $t = t_0$, which can be obtained experimentally in principle.

The information on $\{\theta_\alpha\}_{\alpha=1}^L$ and $\{\rho_\alpha\}_{\alpha=0}^L$ allows us to reconstruct $\tilde{H}_{SE}$ so that it satisfies Eq. (S48). Here is a lemma that shows a property the desired matrix should have:

Lemma 6. *A set of real numbers $\{\theta_\alpha\}_{\alpha=1}^L$, a set of linear operators $\{\rho_\alpha\}_{\alpha=0}^L \subset \mathfrak{B}(\mathcal{H}_S \otimes \mathcal{H}_A)$, and a Hermitian matrix $H \in \mathfrak{B}(\mathcal{H}_A)$ satisfy the following equations:*

$$\begin{aligned}&[H, \rho_0] = 0\\ &[H, \rho_\alpha] = -\theta_\alpha\rho_\alpha, \quad (1 \le \forall\alpha \le L,)\end{aligned} \tag{S52}$$

if and only if $\rho_{SA}(t) \in \mathfrak{B}(\mathcal{H}_S \otimes \mathcal{H}_A)$ given by Eq. (S49) satisfies

$$i\frac{d}{dt}\rho_{SA}(t) = [H, \rho_{SA}(t)]. \tag{S53}$$

In Eqs. (S52) and (S53), I_S is omitted for simplicity; that is, H in these equations means $H \otimes I_S$.

(Proof)
Simply taking the time derivative of Eq. (S49) and then using Eq. (S52) lead to Eq. (S53) to prove the *only if* part. The *if* part can be shown by substituting Eq. (S49) into Eq. (S53):

$$\begin{aligned}[H, \rho_0] + \sum_{\alpha=1}^{L}\Big\{&\left([H, \rho_\alpha] - \theta_\alpha\rho_\alpha\right)e^{i\theta_\alpha t}\\ &+ \left([H, \rho_\alpha^\dagger] + \theta_\alpha\rho_\alpha^\dagger\right)e^{-i\theta_\alpha t}\Big\} = 0.\end{aligned} \tag{S54}$$

Equation (S52) follows if we apply $\lim_{T\to+\infty}\frac{1}{T}\int_0^T dte^{i\theta_{\alpha'}t}$ to Eq. (S54). ■

Lemma 6 assures that any Hermitian matrix H satisfying Eq. (S52) can be identified as an $\tilde{H}_{SE}$ (taking its transpose H^T). The existence of such a matrix H, i.e., a Hermitian matrix that satisfies Eq. (S52), is guaranteed by Corollary 4. Therefore, our task now is to find a specific form of H, given the information on $\rho_{SA}(t)$.

To this end, we fix an orthonormal basis of all Hermitian operators $\{A_j\}_{j=1}^{d_S^4\tilde{d}_E^2} \subset \mathfrak{B}(\mathcal{H}_S \otimes \mathcal{H}_A)$, each of which has the form $A_j = B_j \otimes I_S$, where $\{B_j\}_{j=1}^{d_S^2\tilde{d}_E^2} \subset \mathfrak{B}(\mathcal{H}_A)$ (Recall that our target Hamiltonian has the form $H_A \otimes I_S$ and ρ_{A_1S} is maximally entangled.). We also let $\{\epsilon_{jkl}\}_{jkl}$ $(j, k, l \in \{1, 2, ..., d_S^4\tilde{d}_E^2\})$ denote a set of structure constants of the Lie algebra generated by $\{iA_j\}_{j=1}^{d_S^4\tilde{d}_E^2}$, i.e.,

$$[iA_j, iA_k] = \sum_{l=1}^{d_S^4\tilde{d}_E^2}\epsilon_{jkl}iA_l. \tag{S55}$$

Since the basis set $\{A_j\}_{j=1}^{d_S^4\tilde{d}_E^2}$ is a basis of all Hermitian operators on $\mathcal{H}_S \otimes \mathcal{H}_A$, we can uniquely expand the following operators in terms of $\{A_j\}$:

$$\begin{aligned}\rho_0 &= \sum_{j=1}^{d_S^4\tilde{d}_E^2}u_jA_j,\\ \rho_\alpha + \rho_\alpha^\dagger &= \sum_{j=1}^{d_S^4\tilde{d}_E^2}v_j^{(\alpha)}A_j,\\ i\left(\rho_\alpha - \rho_\alpha^\dagger\right) &= \sum_{j=1}^{d_S^4\tilde{d}_E^2}w_j^{(\alpha)}A_j,\end{aligned} \tag{S56}$$

where $\{u_j\}_{j=1}^{d_S^4\tilde{d}_E^2}$, $\left\{v_j^{(\alpha)}\right\}_{j=1}^{d_S^4\tilde{d}_E^2}$, and $\left\{w_j^{(\alpha)}\right\}_{j=1}^{d_S^4\tilde{d}_E^2}$ are all real constants for all α. Similarly, an arbitrary Hermitian operator H on $\mathcal{H}_A$ can be written as

$$H\otimes I_S=\sum_{j=1}^{d_S^2\tilde{d}_E^2}h_jA_j, \tag{S57}$$

with real constants $\{h_j\}_{j=1}^{d_S^2\tilde{d}_E^2}$. Then, a necessary and sufficient condition for H, $\{\theta_\alpha\}_{\alpha=1}^L$, and $\{\rho_\alpha\}_{\alpha=0}^L$ to satisfy Eq. (S52) is that $\{h_j\}_{j=1}^{d_S^2\tilde{d}_E^2}$ satisfies the system of linear equations with $1\le l\le d_S^4\tilde{d}_E^2$ and $1\le\alpha\le L$:

$$\begin{aligned}\sum_{j=1}^{d_S^2\tilde{d}_E^2}\left(\sum_{k=1}^{d_S^4\tilde{d}_E^2}\epsilon_{jkl}u_k\right)h_j&=0,\\ \sum_{j=1}^{d_S^2\tilde{d}_E^2}\left(\sum_{k=1}^{d_S^4\tilde{d}_E^2}\epsilon_{jkl}v_k^{(\alpha)}\right)h_j&=\theta_\alpha w_k^{(\alpha)},\\ \sum_{j=1}^{d_S^2\tilde{d}_E^2}\left(\sum_{k=1}^{d_S^4\tilde{d}_E^2}\epsilon_{jkl}w_k^{(\alpha)}\right)h_j&=-\theta_\alpha v_k^{(\alpha)}.\end{aligned} \tag{S58}$$

The system of linear equations (S58) has at least one solution, as for experimentally obtained $\{\theta_\alpha\}_{\alpha=1}^L$ and $\{\rho_\alpha\}_{\alpha=0}^L$ there must exist H that satisfies Eq. (S52). In fact, the solution to the above equations, Eqs. (S58), is not unique, since there are $d_S^2\tilde{d}_E^2$ unknown parameters in $d_S^4\tilde{d}_E^2$ equations; there are at most d_S^2L independent solutions. By solving Eqs. (S58), we can derive a set of independent Hermitian operators corresponding to the set of their independent solutions. As we have mentioned above, for all such Hermitian operators H, its transpose H^T can be a legitimate Hamiltonian $\tilde{H}_{SE}$ describing the dynamics of our triple, $\left(\tilde{d}_E,|\Upsilon_{SA_1}\rangle\otimes|\Upsilon_{EA_2}\rangle,\tilde{H}_{SE}\right)$. Hence, the mission completed. :-)

IV. A PROTOCOL FOR STATE-STEERING TOWARDS THE FULFILLMENT OF THE ME CONDITION

As we have seen in Sec. II, as long as our access is limited to the principal and ancillary systems S and A, the best we can do (from quantum control perspective) is to determine an equivalence class of triples in the form of $(d_E,|\Psi_{SEA}\rangle,H_{SE})$. The information on H_{SE} thereby obtained is sufficient to exploit environment E as a resource for quantum computing by actively controlling it via system S. In order for this Hamiltonian identification to work out, the state $|\Psi_{SEA}\rangle$ has to satisfy the ME condition in our scenario.

Therefore, we need a method to steer any given state on $\mathcal{H}_S\otimes\mathcal{H}_E$ towards such a state that fulfills the ME condition. In this section, we present a protocol for this task allowing us to append an extra (ancillary) system $\mathcal{H}_A$.

As we have mentioned in Sec I, it can be taken for granted that the whole state on $\mathcal{H}_S\otimes\mathcal{H}_E$ is pure at the beginning. In addition, this initial state on $\mathcal{H}_S\otimes\mathcal{H}_E$ is assumed to be the same (fixed), but perhaps an unknown, state after equilibration. We set $t=0$ when each iteration of the protocol starts. The protocol proceeds by iterating a block of steps that consist of four major elements, namely the SWAP operation [3] between S and (a subsystem of) A, the time evolution driven by the Hamiltonian H_{SE}, the local filtering operation $\mathcal{F}_{\rm LF}^A$ on A, and state tomography on S and A. The number of iterations of the block is indexed by C.

Since we need to perform state tomography at sufficiently high frequency, which we will specify later, and there are non-deterministic operations $\mathcal{F}_{\rm LF}^A$, the protocol needs to be iterated (ideally infinitely) many times by resetting the entire state and the clock.

A quantity ΔE_{SA}, which plays a central role in designing the protocol, is defined as a functional of state on $\mathcal{H}_S\otimes\mathcal{H}_A$ as

$$\Delta E_{SA}:=S(\rho_{SA})-S(\rho_A)+\log d_S, \tag{S59}$$

where $S(\rho)=-{\rm Tr}\rho\ln\rho$ is the von Neumann entropy of state ρ [15–17].

The local filtering operation $\mathcal{F}_{\rm LF}^A$ is given as $\mathcal{F}_{\rm LF}^A(\rho)=F_{\rm LF}\rho F_{\rm LF}$, where $F_{\rm LF}:=\sqrt{\lambda_{\min}\cdot\rho_A^{-1}}$, where ρ_A^{-1} and $\lambda_{\min}$ are the inverse of ρ_A on its support and the smallest eigenvalue of ρ_A. The local filtering operation succeeds with probability $\lambda_{\min}\cdot{\rm rank}\rho_A$. When the local filtering fails, we abort the present protocol and restart it from the beginning.

The protocol proceeds as follows:

Step 0. At $t=0$, i.e., before any iteration of the following steps, there is ancillary system A, thus $\dim\mathcal{H}_A=0$. Set the counter $C=0$.

Step 1. Prepare a standard MES on a pair of new d_S-dimensional ancillary systems, $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}$. Apply $SWAP_{SA_1}$ on $\mathcal{H}_S\otimes\mathcal{H}_{A_1}$ and relabel the resulting group of AA_1A_2 as a new A system.

Step 2. Apply the local filtering operation $\mathcal{F}_{\rm LF}^A$ on $\mathcal{H}_A$ and increase C by one and call this time t_C.

Step 3. Evaluate ΔE_{SA} in Eq. (S59) through state tomography on SA and define ϵ_C as

$$\epsilon_C:=\frac{1}{2}\sup\left\{\Delta E_{SA}\left(t\right)\ \mid t\in[t_C,t_C+\Delta t]\right\} \tag{S60}$$

during $[t_C,t_C+\Delta t_C]$.

Step 4. Terminate the protocol if $\epsilon_C=0$; otherwise, let the SE system evolve until ΔE_{SA} becomes larger than (or equal to) ϵ_C and go back to Step 1.

As mentioned earlier, because state tomography is involved in Step 3, the evaluation of ΔE_{SA} can be achieved by iterating all the preceding steps many times. Having obtained ΔE_{SA} as a function of time over $[t_C, t_C + \Delta t]$, we set ϵ_C to be a threshold that can be attained within this time period. The factor of 1/2 in Eq. (S60) is chosen merely for convenience to define an achievable threshold.

How long can Δt be? It is the time length, within which we obtain ΔE_{SA} as a function of time by repeating state tomography. As long as our active controls, such as SWAP and $\mathcal{F}^A_{\mathrm{LF}}$, can be performed in an error-free manner, as we assume in this study, Δt can be arbitrary. In order to minimise the overall time length, the shorter the Δt is the better; however, if Δt is too short the local filtering would succeed with only a very small probability. Therefore, a more realistic length of Δt would be the period of 'oscillation' of ΔE_{SA}, which means that the value of Δt may vary from time to time. Nevertheless, this strategy should work, since Δt only needs to have an appropriate ϵ_C, and its precise, or best, value does not have to be known a priori.

An important observation follows as a theorem:

Theorem 3. *If $d_E := \dim \mathcal{H}_E$ is finite, the protocol halts when $C = d_E$ at latest, i.e., before $t = d_E \Delta t$. In other words, there exists a natural number $K \leq d_E$ such that $\Delta E_{SE}(t) = 0$ for all $t \in [t_K, t_K + \Delta t]$, or equivalently, $\epsilon_K = 0$.*

Let us first see that ΔE_{SE} at time t is equal to the amount of the increment of entanglement between $\mathcal{H}_A$ and $\mathcal{H}_S \otimes \mathcal{H}_E$, which is induced when Step 1 of the protocol is performed at t.

Lemma 7. *Suppose that the joint system SEA at time t in the protocol is described by a state $|\Psi_{SEA}\rangle$, and $|\Psi^{in}_{SEA}\rangle$ and $|\Psi^{out}_{SEA}\rangle$ are states on $\mathcal{H}_S \otimes \mathcal{H}_E \otimes \mathcal{H}_A \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ given as*

$$\begin{aligned} |\Psi^{in}_{SEA}\rangle &:= |\Psi_{SEA}\rangle \otimes |\Upsilon_{A_1A_2}\rangle, \\ |\Psi^{out}_{SEA}\rangle &:= SWAP_{SA_1} |\Psi^{in}_{SEA}\rangle \\ &= |\Psi_{AA_1E}\rangle \otimes |\Upsilon_{SA_2}\rangle, \end{aligned} \tag{S61}$$

where $|\Upsilon_{A_1A_2}\rangle$ and $|\Upsilon_{SA_2}\rangle$ are written in the standard form of d_S-dimensional maximally entangled states on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $\mathcal{H}_S \otimes \mathcal{H}_{A_2}$, respectively. Then, for $\rho_{SA} := \mathrm{Tr}_E P(|\Psi_{SEA}\rangle)$,

$$\Delta E_{SE} = E(|\Psi^{out}_{SEA}\rangle) - E(|\Psi^{in}_{SEA}\rangle, \tag{S62}$$

where $E(|\Psi\rangle)$ is the amount of entanglement with respect to the partition between $\mathcal{H}_S \otimes \mathcal{H}_E$ and $\mathcal{H}_A \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ [3, 12, 18, 19].

(**Proof**)

$$\begin{aligned} &E(|\Psi^{out}_{SEA}\rangle) - E(|\Psi^{in}_{SEA}\rangle) \\ =&E(|\Psi_{AA_1E}\rangle \otimes |\Upsilon_{SA_2}\rangle) - E(|\Psi_{SEA}\rangle \otimes |\Upsilon_{A_1A_2}\rangle) \\ =&E(|\Psi_{AA_1E}\rangle \otimes |\Upsilon_{SA_2}\rangle) - E(|\Psi_{SEA}\rangle) \\ =&S(\rho_{AA_1} \otimes \rho_{mix}) - S(\rho_A) \\ =&S(\rho_{SA}) - S(\rho_A) + \log d_S \\ =&\Delta E_{SE}. \end{aligned}$$

In the RHS of the fourth equation, ρ_{mix} is a completely mixed state on $\mathcal{H}_S$. ■

Note that ΔE_{SE} is evaluated with respect to a state before the SWAP between S and A_1. During Step 3, the state $\rho_A := \mathrm{Tr}_{SE} P(|\Psi_{SEA}\rangle)$ does not change, and stays as a projector on $\mathcal{H}_A$ (up to a proportionality constant) as a result of $\mathcal{F}^A_{\mathrm{LF}}$. Thus $|\Psi^{in}_{SEA}\rangle$ is a MES on their support, and if $\Delta E_{SE} > 0$, the Schmidt rank of $|\Psi^{out}_{SEA}\rangle$ is strictly greater than that of $|\Psi^{in}_{SEA}\rangle$. This means that $\mathrm{rank}\rho^{out}_A - \mathrm{rank}\rho_A \geq 1$; namely, applying the $SWAP_{SA_1}$ in the Step 1 increases the rank of the state on A by at least 1.

Theorem 3 can now be proven with these lemmas and facts.

(Proof of Theorem 3)
Note that the state on the joint system SEA is always pure throughout the protocol as long as all the local filtering operations succeed. Since the local filtering operation $\mathcal{F}^A_{\mathrm{LF}}$ preserves the Schmidt rank of $|\Psi_{SEA}\rangle$ with respect to the partition $A|SE$, the execution of Step 1 increases the Schmidt rank of $|\Psi_{SEA}\rangle$ by at least 1 (as we have seen above). The Schmidt rank of $|\Psi_{SEA}\rangle$ with respect to the same partition is obviously smaller than d_E. Thus C is no greater than d_E and the statement of the theorem follows. ■

The following lemma shows another, more intuitive, meaning of ΔE_{SE}:

Lemma 8. *The following equation holds for ΔE_{SE}:*

$$\Delta E_{SE} = D(\rho_{SE} \| \rho_S \otimes \rho_E) + D(\rho_S \| \rho_{mix}), \tag{S63}$$

where $D(\rho\|\sigma)$ is the relative entropy defined as $D(\rho\|\sigma) := \mathrm{Tr}\rho \log \rho - \mathrm{Tr}\rho \log \sigma$ [3, 4, 16, 17, 20, 21].

(**Proof**)

$$\begin{aligned} &\Delta E_{SE} \\ =&S(\rho_E) - S(\rho_{SE}) + \log d_S \\ =&S(\rho_E) + S(\rho_S) - S(\rho_{SE}) - S(\rho_S) - \mathrm{Tr}\rho_S \log \rho_{mix} \\ =&I_\rho(S\|E) + D(\rho_S|\rho_{mix}) \\ =&D(\rho_{SE}\|\rho_S \otimes \rho_E) + D(\rho_S\|\rho_{mix}), \end{aligned}$$

where $I_\rho(S\|E)) := S(\rho_S) + S(\rho_E) - S(\rho_{SE})$ is the mutual information between S and E, and we use the formula $I_\rho(S\|E) = D(\rho_{SE}\|\rho_S \otimes \rho_E)$ [3] in the fourth equation. ■

This lemma guarantees that when ΔE_{SE} is small, ρ_{SE} is close to $\rho_S \otimes \rho_E$ and ρ_S is almost completely mixed. We can also show the following:

Lemma 9. *When $\rho_{SE} = \rho_S \otimes \rho_E$, there exists a decomposition of system A into two distinct subsystems A_1 and A_2; that is, $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$. Accordingly, a (pure) state $|\Psi_{SEA}\rangle$ can be written as a product of pure states $|\Phi_{SA_1}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_{A_1}$ and $|\Phi_{EA_2}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_{A_2}$ such that*

$$|\Psi_{SEA}\rangle = |\Phi_{SA_1}\rangle \otimes |\Phi_{EA_2}\rangle, \tag{S64}$$

where $|\Phi_{SA_1}\rangle$ and $|\Phi_{EA_2}\rangle$ satisfies $\mathrm{Tr}_{A_1} P(|\Phi_{SA_1}\rangle) = \rho_S$ and $\mathrm{Tr}_{A_2} P(|\Phi_{EA_2}\rangle) = \rho_E$, respectively. In particular, when $\rho_S = \rho_{mix}$, $|\Phi_{SA_1}\rangle$ can be chosen as a standard maximally entangled state, i.e., $|\Upsilon_{SA_1}\rangle = d_S^{-1/2} \sum_{i=1}^{d_S} |i_S\rangle|i_{A_1}\rangle$.

(**Proof**)
Consider any decomposition of A into subspaces A_1' and A_2' such that $\mathcal{H}_A = \mathcal{H}_{A_1'} \otimes \mathcal{H}_{A_2'}$ and $\dim \mathcal{H}_{A_1'} = \dim \mathcal{H}_S$. Then there exist the purifications $|\Phi'_{A_1E}\rangle$ and $|\Phi'_{EA_2}\rangle$ of ρ_S and ρ_E in each subspace, i.e., $\mathrm{Tr}_{A_1} P(|\Phi'_{SA_1}\rangle) = \rho_S$ and $\mathrm{Tr}_{A_2} P(|\Phi'_{EA_2}\rangle) = \rho_E$, respectively. As $|\Psi_{SEA}\rangle$ and $|\Phi_{A_1'S}\rangle \otimes |\Phi_{A_2'E}\rangle$ have the same reduced density matrix on $\mathcal{H}_S \otimes \mathcal{H}_E$, there exists a unitary operator U_A on $\mathcal{H}_A$ such that [3, 4]

$$|\Psi_{SEA}\rangle = U_A |\Phi_{A_1'S}\rangle \otimes |\Phi'_{A_2'E}\rangle. \tag{S65}$$

Defining $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} := U_A \mathcal{H}_{A_1'} \otimes \mathcal{H}_{A_2'}$ proves the lemma. ■

With Theorem 3 and Lemmas 8 and 9, we finally arrive at the following theorem:

Theorem 4. *Let K be the index of the counter C when the protocol halts. Then, for all $t_\infty \geq t_K + \Delta t$, $(d_E, |\Psi_{SEA}(t_K)\rangle, H_{SE})$ satisfies the ME condition for $[t_K, t_\infty)$, where $|\Psi_{SEA}(t_K)\rangle$ is the state on SEA after the K-th (successful) application of the local filtering operation $\mathcal{F}^A_{\mathrm{LF}}$ in Step 2 at time t_K.*

(**Proof**)
By the construction of the protocol, $\Delta E_{SE}(t) = 0$ for all $t \in [t_K, t_K + \Delta t]$. By Lemma 8, $\rho_{SE}(t) = \rho_S(t) \otimes \rho_E(t)$ and $\rho_S(t)$ is a maximally mixed state for all $t \in [t_K, t_k + \Delta t]$. Then, due to Lemma 9, for all $t \in [t_K, t_K + \Delta t]$, there exist Hilbert spaces $\mathcal{H}_{A_1}(t)$ and $\mathcal{H}_{A_2}(t)$ such that $\mathcal{H}_A = \mathcal{H}_{A_1}(t) \otimes \mathcal{H}_{A_2}(t)$ and $|\Psi_{SEA}(t)\rangle$ satisfies

$$\mathrm{Tr}_E P(|\Psi_{SEA}(t)\rangle) = P\left(|\Upsilon_{A_1(t)S}\rangle\right) \otimes \rho_{A_2(t)}, \tag{S66}$$

where $|\Upsilon_{A_1(t)S}\rangle$ is a maximally entangled state on $\mathcal{H}_S \otimes \mathcal{H}_{A_1}(t)$. Moreover, $\rho_{A_2(t)}(t)$ is a projector, because of the local filtering operation in Step 2. Therefore, $(d_E, |\Psi_{SEA}(t_K)\rangle, H_{SE})$ fulfills the ME condition for $[t_K, t_K + \Delta t]$. Corollary 2 then lets us finish the proof of the theorem. ■

We have seen that steering the entire system of A, S, and E towards one that satisfies the ME condition can be achieved within a finite time (Theorems 3 and 4). Once this state-steering has been done, the identification of the equivalence class by probing only systems S and A can be performed (Sec. III). Therefore, we can always determine the equivalence class of a triple associated with a finite-dimensional environment, which is essential in identifying the Hamiltonian H_{SE}, starting from an arbitrary state on the system SEA.

In the present study, we have assumed that the state tomography on SA can be performed perfectly. The feasibility of the presented protocol for state-steering depends on this assumption: in order to complete the task, we need to check whether $\Delta E_{SE}(t)$ is exactly 0 for all $t \in [t_C, t_C + \Delta t]$. Although we have focused on the theoretical aspect of our tomographic scheme for a 'not directly touchable system', which is quite remarkable in its own right, any quantum operation, including state tomography, is always fraught with the effect of unpredictable noise in reality. Therefore, from a pragmatic point of view, we need to modify the protocol and evaluate errors that may occur in the equivalent class identification. Since the analysis of errors in the protocol is beyond the scope of this paper, we leave it as a future project [22].

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